

## Lecture 1 (30/9/2014)

### Linear Analysis.

The interest of Linear Analysis lies in normed spaces. Normed spaces have two very nice properties: they are vector spaces, and at the same time metric spaces, with a particular metric induced by the norm, which in turn happens to have some nice properties.

This is how we will treat normed spaces: as vector spaces with a nice metric. The open sets of a normed space will be the ones that are open with respect to this special metric. So: all the properties that have to do with openness, such as convergence and completeness, will be determined by this metric induced by the norm!

Def: Let  $X$  be a set. A metric on  $X$  is a function  $d: X \times X \rightarrow \mathbb{R}$ , s.t.

(i)  $d(x, y) \geq 0 \quad \forall x, y \in X, d(x, y) = 0 \iff x = y$ .

(ii)  $d(x, y) = d(y, x), \quad \forall x, y \in X$ .

(iii)  $d(x, z) \leq d(x, y) + d(y, z), \quad \forall x, y, z \in X$  (triangle inequality).

Def: A metric space  $(X, d)$  is a set  $X$ , equipped with a metric  $d$ .

Def: A vector space over a field  $\mathbb{F}$  (for us,  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ), is a set  $X$ , together with two operations,  $+ : X \times X \rightarrow X$  (addition) and  $\cdot : \mathbb{F} \times X \rightarrow X$  (scalar multiplication), s.t.:

(i)  $x + (y+z) = (x+y)+z$ ,  $\forall x, y, z \in X$ .

(ii)  $x+y = y+x$ ,  $\forall x, y \in X$ .

(iii) There exists an element  $0 \in X$ , called the zero vector, s.t.  $0+x=x$ ,  $\forall x \in X$ .

(iv)  $\forall x \in X$ , there exists an element  $-x \in X$ , s.t.  $x+(-x)=0$ .

(v)  $\lambda_1 \cdot (\lambda_2 \cdot x) = (\lambda_1 \lambda_2) \cdot x$ ,  $\forall \lambda_1, \lambda_2 \in F$ .

(vi)  $1 \cdot x = x$ ,  $\forall x \in X$  (where 1 is the multiplicative identity element of  $F$ ;  
i.e. the number 1 in  $\mathbb{R}$  or  $\mathbb{C}$ ).

(vii)  $\lambda \cdot (x+y) = \lambda \cdot x + \lambda \cdot y$ ,  $\forall \lambda \in F$ ,  $\forall x, y \in X$ .

(viii)  $(\lambda_1 + \lambda_2) \cdot x = \lambda_1 \cdot x + \lambda_2 \cdot x$ ,  $\forall \lambda_1, \lambda_2 \in F$ ,  $\forall x \in X$ .

ex: •  $\mathbb{R}^n$  : a vector space over  $\mathbb{R}$ .

•  $\mathbb{C}^n$  : a vector space over  $\mathbb{C}$ .

- $C(K) := \{f: K \rightarrow \mathbb{R} \text{ continuous}\}$  (where  $K$  is any subset of  $\mathbb{R}$ ):

a vector space over  $\mathbb{R}$ .

- $c_0$ : the set of sequences  $(x_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}$  s.t.  $x_n \xrightarrow{n \rightarrow \infty} 0$ :

a vector space over  $\mathbb{R}$ .

- $c_00$ : the set of sequences in  $\mathbb{R}$  with finitely many non-zero terms:

a vector space over  $\mathbb{R}$ .

- $\ell^p := \{(x_n)_{n \in \mathbb{N}} \text{ in } \mathbb{R} : \sum_{n=1}^{+\infty} |x_n|^p < +\infty\}, \text{ if } 1 \leq p < +\infty,$

$$\ell^\infty := \{(x_n)_{n \in \mathbb{N}} \text{ in } \mathbb{R} : \sup_{n \in \mathbb{N}} |x_n| < +\infty\}$$

$\downarrow$   
(the set of  
bounded sequences)

vector spaces over  $\mathbb{R}$ .

$$\bullet L^p := \left\{ f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\} : \int_{\mathbb{R}^n} |f(x)|^p dx < +\infty \right\}, \text{ if } 1 \leq p < +\infty,$$

or  $\mathbb{C} \cup \{\infty\}$

(the set of functions on  $\mathbb{R}^n$  that are essentially bounded.)

This practically means that

if they are unbounded only on a set of measure 0, we still consider them

as bounded)

These are vector spaces over  $\mathbb{R}$  (or  $\mathbb{C}$ )

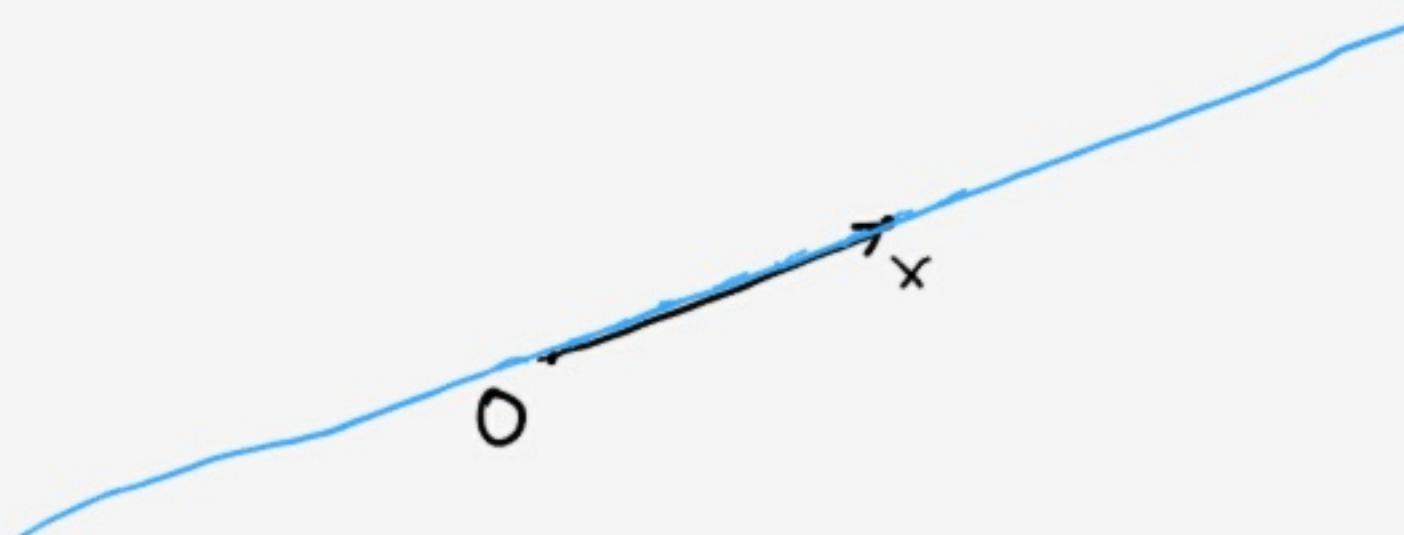
$$L^\infty := \left\{ f: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\} \text{ measurable (w.r.t. the Lebesgue measure), s.t.} \begin{array}{l} \text{the essential supremum of } f \text{ is } < +\infty \\ \text{(the essential supremum of } f \text{ is the infimum of all } a \in \mathbb{R} \cup \{\pm\infty\} \text{,} \\ \text{or } \mathbb{C} \cup \{\infty\} \end{array} \right\}$$

s.t. the Lebesgue measure of the set of  $x \in \mathbb{R}^n$  with  $|f(x)| \geq a$  is 0).

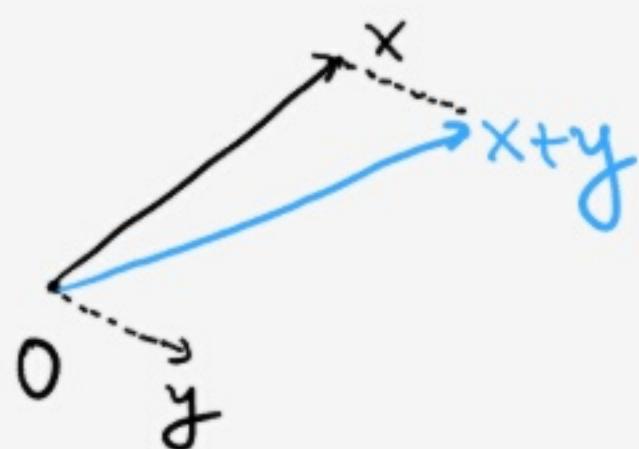


Let  $X$  be a vector space. The fact that addition and scalar multiplication have images in  $X$  means that:

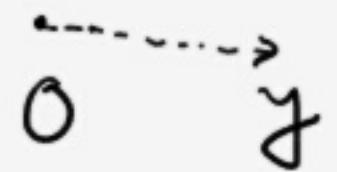
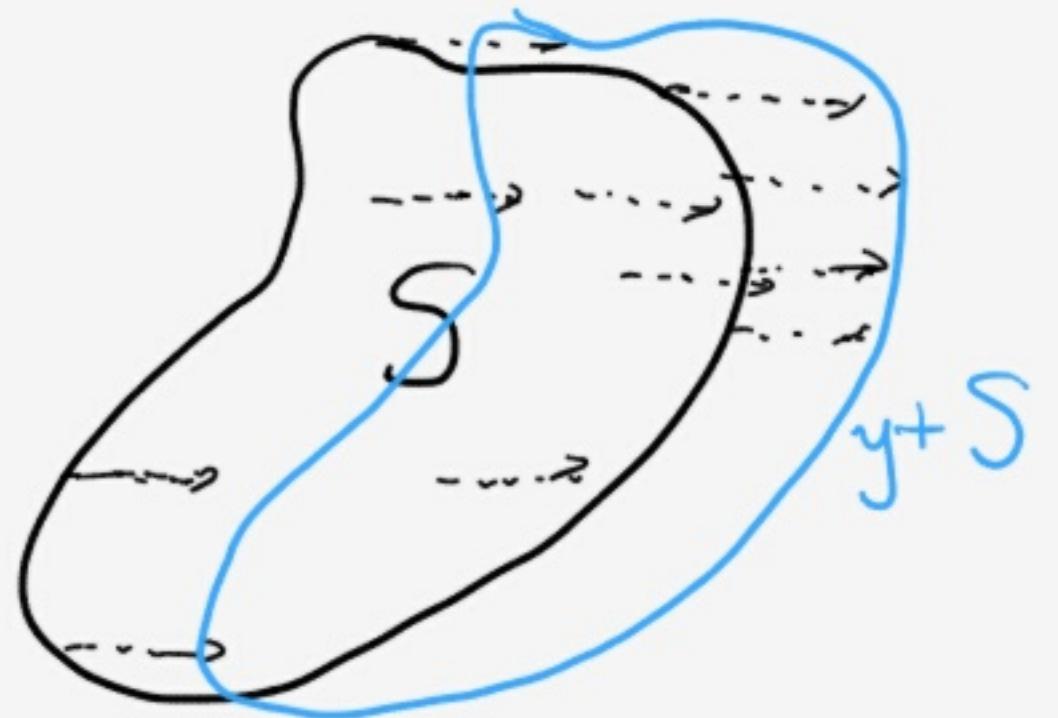
- (i) If  $x \in X$ , then  $\lambda x \in X \ \forall \lambda \in \mathbb{F}$ , i.e.: "the line connecting  $x$  with  $0$  lies in  $X$ "



- (ii) If  $x \in X$ , then  $x+y \in X \ \forall y \in X$ , i.e. "the translation of  $x$  by a vector in  $X$  lies in  $X$ ".



(iii) If  $S \subseteq X$  and  $y \in X$ , then  $y + S \subseteq X$  (where  $y + S := \{y + s : s \in S\}$ ), i.e.  
"the translation of  $S$  by a vector in  $X$  lies in  $X$ ".



Def: Let  $X$  be a vector space over  $\mathbb{F}$ . Let  $E$  be a subset of  $X$ . The linear span of  $E$ ,

which we denote by  $\text{span } E$ , is the set of all finite linear combinations of elements of  $E$ .

$$(\text{i.e., } \text{span } E := \{ \lambda_1 e_1 + \dots + \lambda_k e_k : k \in \mathbb{N}, e_1, \dots, e_k \in E, \lambda_1, \dots, \lambda_k \in F \}).$$

Def: Let  $X$  be a vector space over  $F$ . A (Hamel) basis of  $X$  is a subset  $E$  of  $X$ , s.t.

(i)  $X = \text{span } E$ ,

(ii)  $E$  is linearly independent (i.e., any finitely many elements of  $E$  are linearly independent).



Do not confuse this with an orthonormal basis  $E$  of a Hilbert space  $H$ .

There,  $H = \overline{\text{span } E}$ , where the closure is taken w.r.t. the norm induced by the inner product.



Every vector space has a (Hamel) basis (this can be proved with Zorn's Lemma)

Def.: Let  $X$  be a vector space. Then, it can be proved that all (Hamel) bases of  $X$  have the same cardinality, which we call the dimension of  $X$ , and we denote by  $\dim X$ .

Def.: Let  $X$  be a vector space over  $\mathbb{F}$  ( $= \mathbb{R}$  or  $\mathbb{C}$ ). A norm on  $X$  is a function  $\|\cdot\| : X \rightarrow \mathbb{R}$ , s.t.

- (i)  $\|x\| \geq 0$   $\forall x \in X$ , and  $\|x\|=0 \iff x=0$ .
- (ii)  $\|\lambda x\| = |\lambda| \cdot \|x\|$ ,  $\forall \lambda \in \mathbb{F}$ .
- (iii)  $\|x+y\| \leq \|x\| + \|y\|$ ,  $\forall x, y \in X$  (triangle inequality).

Def: A normed space  $(X, \|\cdot\|)$  is a vector space  $X$ , equipped with a norm  $\|\cdot\|$ .

ex: •  $(l^p, \|\cdot\|_p)$ ,  $1 \leq p \leq +\infty$ , where, if  $x = (x_n)_{n \in \mathbb{N}}$  in  $l^p$ ,  $\|x\|_p = \left( \sum_{n=1}^{+\infty} |x_n|^p \right)^{1/p}$ .

•  $(l^\infty, \|\cdot\|_\infty)$ , where, if  $x = (x_n)_{n \in \mathbb{N}}$  in  $l^\infty$ ,  $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$ .

•  $(L^p, \|\cdot\|_p)$ ,  $1 \leq p \leq +\infty$ , where, if  $f \in L^p$ ,  $\|f\|_p = \left( \int_{x \in \mathbb{R}^n} |f(x)|^p dx \right)^{1/p}$ .

•  $(L^\infty, \|\cdot\|_\infty)$ , where, if  $f \in L^\infty$ ,  $\|f\|_\infty = \text{essential supremum of } f$ .

• Any subspace of a normed space  $(X, \|\cdot\|)$  is a normed space with the same norm.  
$$\left( \begin{array}{l} \text{this is a subset} \\ \text{that is a vector} \\ \text{space} \end{array} \right)$$

for example:  $c_0, c_\infty$  are subspaces of  $\ell^\infty$ , so  $(c_0, \|\cdot\|_\infty), (c_\infty, \|\cdot\|_\infty)$  are normed spaces.

- $C([a,b])$ , the space of cont. functions  $f: [a,b] \rightarrow \mathbb{R}$ , is a subspace of  $L^p$ ,  $\forall 1 \leq p \leq \infty$ , so  $(C([a,b]), \|\cdot\|_p)$  is a norm space,  $\forall 1 \leq p \leq \infty$ .

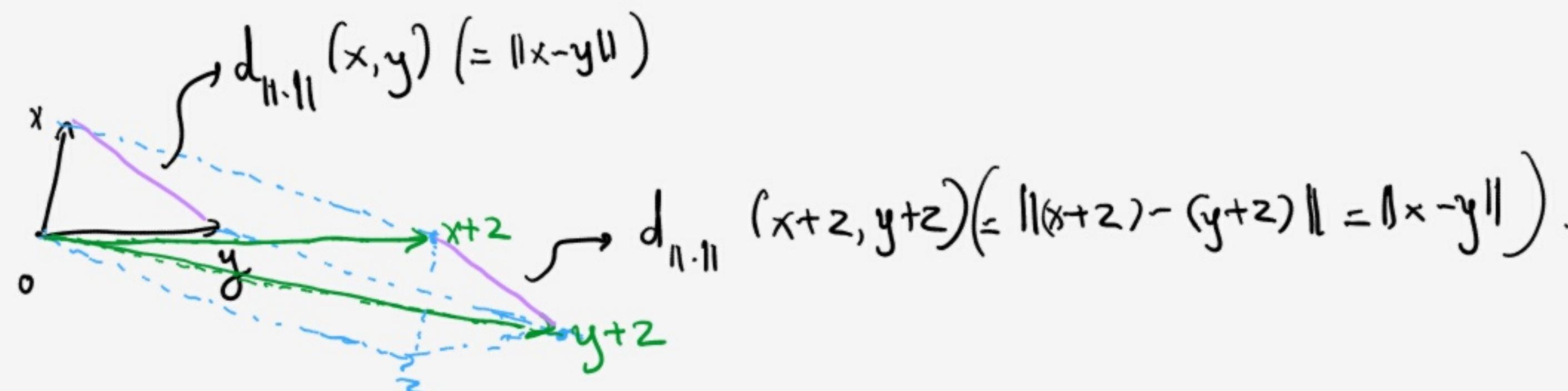
In particular, note that,  $\forall f \in C([a,b]), \|f\|_\infty = \text{ess sup}_{x \in [a,b]} |f(x)| = \sup_{\substack{\text{f cont.} \\ \text{compact}}} \sup_{x \in [a,b]} |f(x)| \leq \max_{[a,b]} |f(x)|$ .

Def: Let  $(X, \|\cdot\|)$  be a normed space. The norm  $\|\cdot\|$  induces a metric  $d_{\|\cdot\|}$  on  $X$ , defined by :  $d_{\|\cdot\|}(x,y) = \|x-y\|, \forall x,y \in X$ .



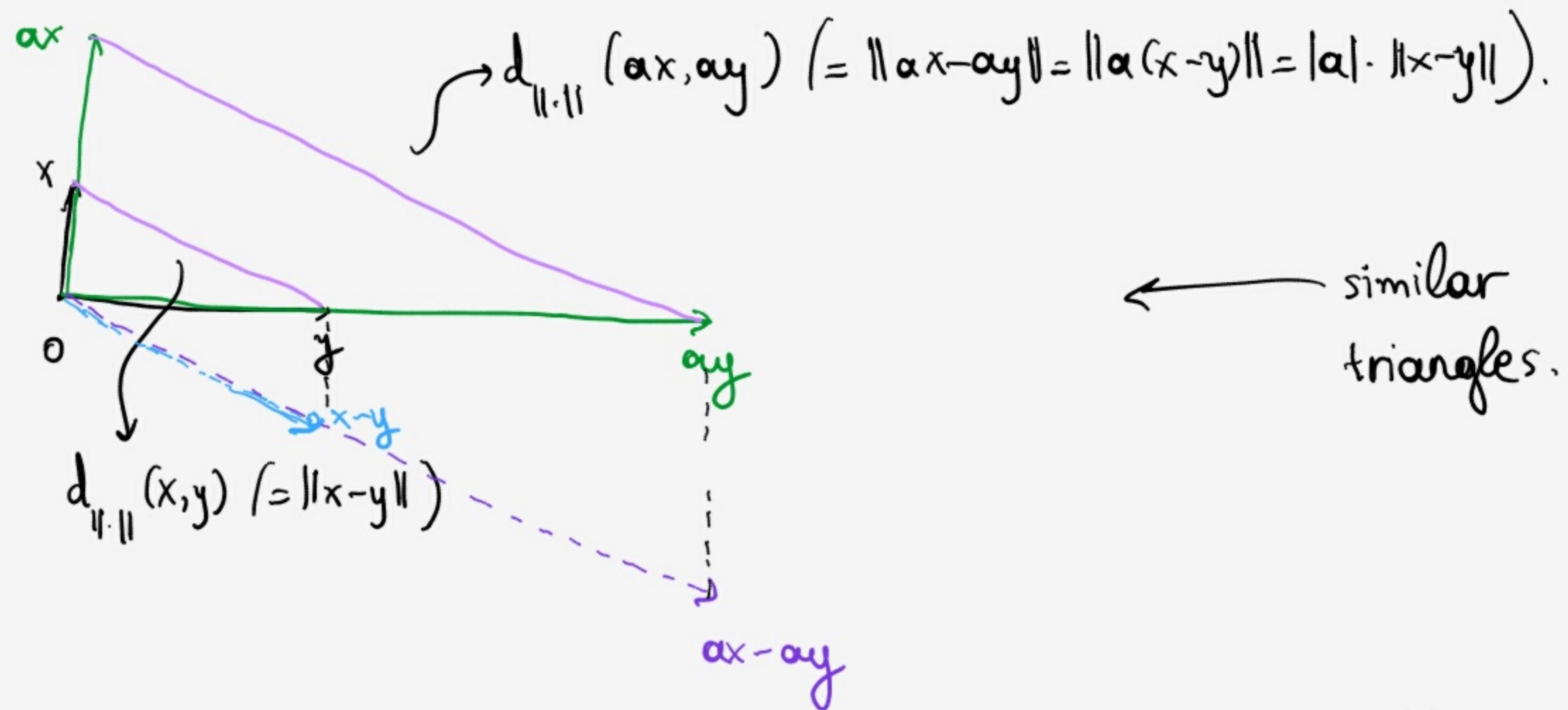
Due to the fact that  $X$  is a vector space, and due to the definition of  $d_{\|\cdot\|}$ , the metric  $d_{\|\cdot\|}$  has two nice properties that not every metric has:

(i) translation invariance:  $d_{\|\cdot\|}(x+z, y+z) = d_{\|\cdot\|}(x, y)$ .



Note that it makes sense to talk about  $x+z, y+z$ , because  $X$  is a vector space and not an arbitrary set.

(ii) homogeneity:  $d_{\|\cdot\|}(\alpha x, \alpha y) = |\alpha| \cdot d_{\|\cdot\|}(x, y)$ , for scalars,  $\forall x, y \in X$ .



Note that it makes sense to talk about  $\alpha x, \alpha y$ , because  $X$  is a vector space and not an arbitrary set.

## Lecture 2 (2/10/2014).

Here, we define open (and thus closed) sets in a normed space.

As we have already mentioned, the open sets will be the ones that are open with respect to the metric induced by the norm.

Therefore, all topological properties of a normed space (i.e. those that can be defined with respect to open sets (such as convergence of sequences)), are fully determined by that particular metric.

Notation: Let  $(X, \|\cdot\|)$  be a normed space. For all  $x \in X, r > 0$ , we denote

$$B(x, r) := \{y \in X : \|y - x\| < r\}.$$

Note that  $B(x, r) = B_{d_{\|\cdot\|}}(x, r)$ ,

where  $B_{d_{\|\cdot\|}}(x, r) := \{y \in X : d_{\|\cdot\|}(y, x) < r\}$ , the open ball in  $(X, d_{\|\cdot\|})$  with centre  $x$  and radius  $r$ .

Def: Let  $(X, \|\cdot\|)$  be a normed space. We say that a subset  $U$  of  $X$  is **open in  $(X, \|\cdot\|)$**  (i.e., open w.r.t. the norm  $\|\cdot\|$ ) if it is open in  $(X, d_{\|\cdot\|})$  (i.e., open w.r.t. the metric  $d_{\|\cdot\|}$ ).

So: 1. In particular,  $B(x, r)$  is open in  $(X, \|\cdot\|)$  (it is an open ball of  $(X, d_{\|\cdot\|})$ , and thus open in  $(X, d_{\|\cdot\|})$ ).

We call  $B(x, r)$  the open ball in  $(X, \|\cdot\|)$  with centre  $x$  and radius  $r$ .

2. By definition,  $U \subseteq X$  is open in  $(X, \|\cdot\|)$  if,  $\forall x \in U$ , there exists some  $B_{d_{\|\cdot\|}}(x, r)$

inside  $U$ ,

i.e.

if,  $\forall x \in U$ ,  $\exists B(x, r) \subseteq U$ .

Def: Let  $(X, \|\cdot\|)$  be a normed space. We say that a subset  $K \subseteq X$  is closed in  $(X, \|\cdot\|)$  if  $X \setminus K$  is open in  $(X, \|\cdot\|)$ .

(In other words,  $K$  is closed in  $(X, \|\cdot\|)$  iff it is closed in  $(X, d_{\|\cdot\|})$ .)

So: the following sets are closed in  $(X, \|\cdot\|)$ :

1.  $\bar{B}(x, r) := \{y \in X : \|y - x\| \leq r\}$  → the closed ball in  $(X, \|\cdot\|)$  with centre  $x$  and radius  $r$ .

2.  $S(x, r) := \{y \in X : \|y - x\| = r\} \rightarrow$  the sphere in  $(X, \|\cdot\|)$  with centre  $x$  and radius  $r$ .

The reason is that  $\bar{B}(x, r) = \bar{B}_{d_{\|\cdot\|}}(x, r)$ , the closed ball in  $(X, d_{\|\cdot\|})$  with centre  $x$  and radius  $r$ ,

and  $S(x, r) = S_{d_{\|\cdot\|}}(x, r)$ , the sphere in  $(X, d_{\|\cdot\|})$  with centre  $x$  and radius  $r$ ,

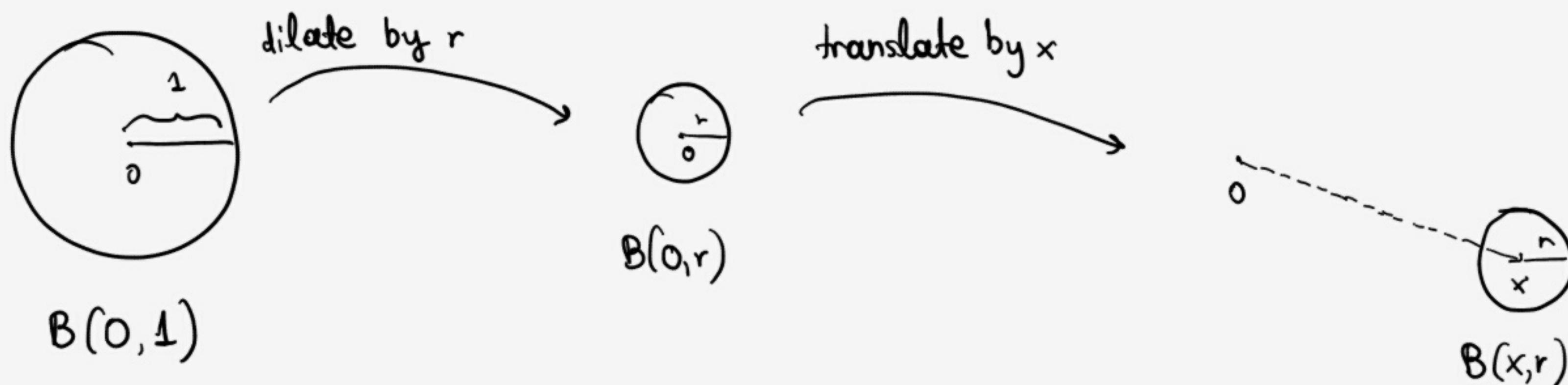
both of which sets are closed in  $(X, d_{\|\cdot\|})$ .

→ for the rest of this lecture, we will investigate properties of open and closed subsets of normed spaces. Many of these properties have to do with open and closed subspaces (i.e., subsets that are also vector spaces).

Def: Let  $(X, \|\cdot\|)$  be a normed space. For all  $x \in X$ , all  $S \subseteq X$  and all scalars  $\alpha$ , we define

- $x + S := \{x + y : y \in S\} \rightarrow$  the translation of  $S$  by  $x$ .
- $\alpha S := \{\alpha y : y \in S\} \rightarrow$  the dilation of  $S$  by  $\alpha$ .

Observations: •  $\forall x \in X, \forall r > 0 : B(x, r) = x + B(0, r) = x + r \cdot B(0, 1)$ .



I.e., it is easy to show that  $rB(0,1) = B(0,r)$  and  $x + B(0,r) = B(x,r)$ . Try it!

- Let  $Y$  be a subspace of  $(X, \|\cdot\|)$  that contains an open ball of  $X$ . Then,  $Y = X$ :

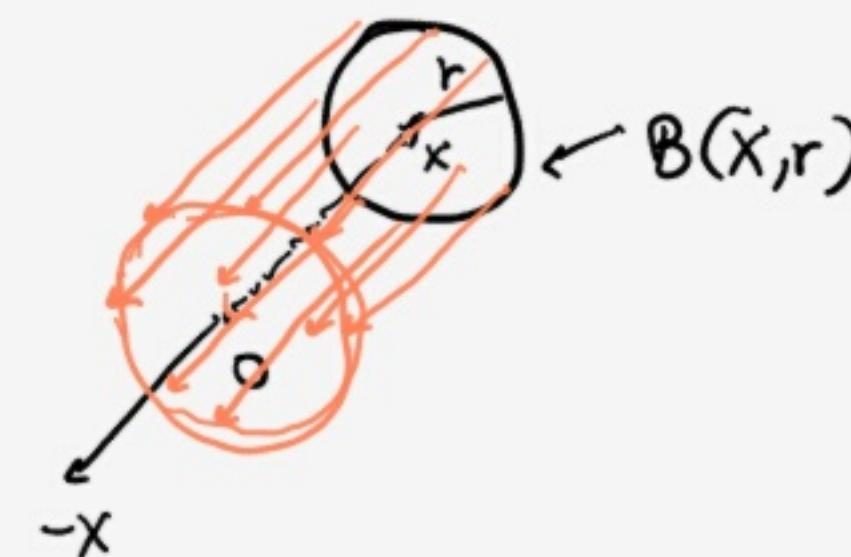
The idea is that, since  $Y$  is a vector space, it will contain the translation of the open ball to  $0$ , i.e. an open ball centered at  $0$ . So,  $Y$  will contain a "vector" in each direction. And since it is a vector space, it will contain all scalar multiples of these vectors (i.e., all lines through  $0$ ), so all vectors.

More precisely:

Suppose that  $B(x, r) \subseteq Y$ .

$\downarrow$   
an open ball of  $(X, \|\cdot\|)$ .

$$B(x, r) \subseteq Y \rightarrow x \in Y \xrightarrow[\text{space}]{Y \text{ vector}} -x \in Y.$$



So, the translation of  $B(x, r)$  by  $-x$ , an element of the vector space  $Y$ , has to lie in  $Y$ . I.e.,  $\begin{matrix} B(x, r) \subseteq Y \\ -x \in Y \end{matrix} \Rightarrow -x + B(x, r) \subseteq Y$ .

But  $-x + B(x, r) = B(0, r)$ , so  $B(0, r) \subseteq Y$ .

So, since  $Y$  is a vector space, all dilations of  $B(0, r)$  lie in  $Y$ , i.e.

$\underbrace{R \cdot B(0, r)}_{\text{all}} \subseteq Y$ ,  $\forall R > 0$ , so  $B(0, M) \subseteq Y$ ,  $\forall M > 0$ . But  $X = \bigcup_{M>0} B(0, M)$  (since each  $x \in X$  has some distance from 0, so it lies in some open ball), so  $X \subseteq Y$ . Also  $Y \subseteq X$ , so  $Y = X$ .

If you find this explanation confusing, another way to see that  $B(0,r) \subseteq Y$  then  $Y = X$  is the following:

Let  $x \in X$ . We will show that  $x \in Y$ . Indeed:

The vector  $\frac{x}{\|x\|}$  has norm 1 ( $\left\| \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|} \cdot \|x\| = 1$ ) (a unit vector with "the same direction" as  $x$ ", i.e. a scalar multiple of  $x$ .)

So, the vector  $r \cdot \frac{x}{\|x\|}$  has norm  $r$  ( $\left\| r \cdot \frac{x}{\|x\|} \right\| = r \cdot \left\| \frac{x}{\|x\|} \right\| = r \cdot 1 = r$ ).

So, the vector  $\frac{r}{2} \cdot \frac{x}{\|x\|}$  has norm  $\frac{r}{2}$ , so  $\frac{r}{2\|x\|} \cdot x \in B(0,r) \rightarrow \frac{r}{2\|x\|} \cdot x \in Y$ .

So, we have found a scalar multiple of  $x$  that lies in  $Y$ . Since  $Y$  is a vector space, all scalar multiples of  $\frac{r}{2\|x\|} \cdot x$  lie in  $Y$ . In particular,

$\frac{2\|x\|}{r} \cdot \left( \frac{r}{2\|x\|} \cdot x \right) \in Y$ , i.e.  $x \in Y$ . So, the idea is to find a vector in  $Y$ , with "the same direction as  $x$ ", i.e. a scalar multiple of  $x$ .

- A normed space has no open subspaces other than itself.

Proof: Let  $(X, \|\cdot\|)$  be a normed space and  $Y$  a subspace of  $X$  that is open in  $(X, \|\cdot\|)$ .

Since  $Y$  open in  $(X, \|\cdot\|)$ , we have that,  $\forall y \in Y, \exists r: B(y, r) \subseteq Y$ .

In particular,  $Y$  contains some open ball of  $X$ , so  $Y = X$ .

- Let  $Y$  be a subspace of  $(X, \|\cdot\|)$  that contains a sphere of  $X$ . Then,  $Y = X$ .

Can you show it?

→ Convergence of sequences in normed spaces:

Convergence of a sequence in a normed space is convergence w.r.t. the metric induced by the norm. More precisely:

Let  $(X, d)$  be a metric space. We say that a sequence  $(x_n)_{n \in \mathbb{N}}$  converges in  $(X, d)$  to some  $x \in X$ , and we denote by  $x_n \xrightarrow{d} x$ , if

$$d(x_n, x) \xrightarrow[n \rightarrow +\infty]{} 0$$

Def: Let  $(X, \|\cdot\|)$  be a normed space. We say that a sequence  $(x_n)_{n \in \mathbb{N}}$  converges in  $(X, \|\cdot\|)$  to some  $x \in X$ , and we denote by  $x_n \xrightarrow{\|\cdot\|} x$ , if  $x_n \xrightarrow{d_{\|\cdot\|}} x$ , i.e.

$$\text{if } \|x_n - x\| \xrightarrow[n \rightarrow +\infty]{} 0$$

→ Characterisation of closed sets via convergence of sequences.

Let  $(X, d)$  be a metric space. Then, a well-known theorem states that

a subset $K$ of $X$ is closed in $(X, d)$	$\iff$	for each sequence $(x_n)_{n \in \mathbb{N}}$ in $K$ , it holds that, if $x_n \xrightarrow{d} x$ , for some $x \in X$ , then $x \in K$ (in other words, all the limit points of $K$ in $X$ lie in $K$ ).
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So, if  $(X, \|\cdot\|)$  is a normed space and  $K \subseteq X$ , then

$$\underline{K \text{ closed in } (X, \|\cdot\|)} \iff \underset{\text{def.}}{K \text{ closed in } (X, d_{\|\cdot\|})}$$

$$\iff \forall (x_n)_{n \in \mathbb{N}} \text{ in } K, \text{ it holds that, if } x_n \xrightarrow{d_{\|\cdot\|}} x$$

for some  $x \in X$ , then  $x \in K$

convergence  
w.r.t.  $\|\cdot\|$  is

convergence  
w.r.t.  $d_{\|\cdot\|}$

$$\iff \forall (x_n)_{n \in \mathbb{N}} \text{ in } K, \text{ it holds that, if } x_n \xrightarrow{\|\cdot\|} x \quad )$$

for some  $x \in X$ , then  $x \in K$

$$\iff \forall (x_n)_{n \in \mathbb{N}} \text{ in } K, \text{ it holds that, if } \|x_n - x\| \xrightarrow{n \rightarrow \infty} 0$$

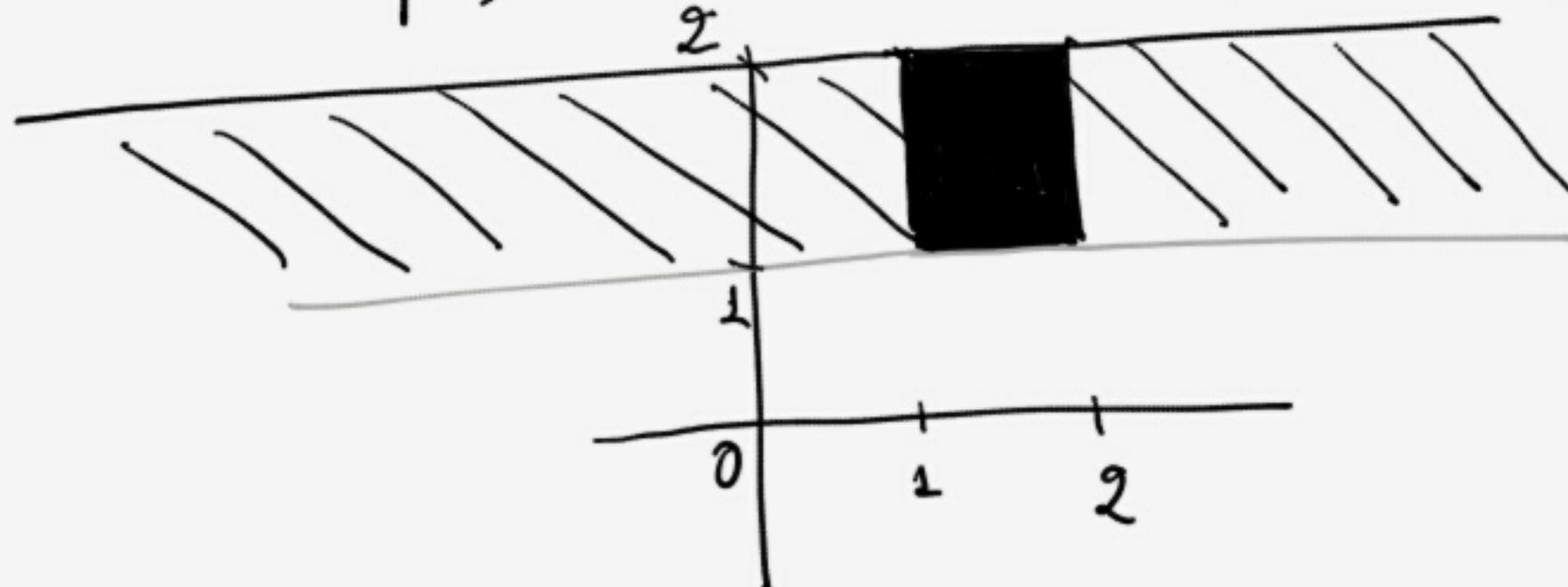
for some  $x \in X$ , then  $x \in K$ .



This characterisation of closedness demonstrates very well the fact that, the larger  $X$  gets, the harder it is for a  $K \subseteq X$  to be closed in  $(X, d)$ .

Indeed, the larger  $X$  is, the larger the potential set of limit points of sequences in  $K$  becomes, so the harder it is for it to lie in the (relatively small) set  $K$ .

For example, the cube  $[1,2] \times [1,2]$  is:



- closed in  $([1,2] \times [1,2], \|\cdot\|_1)$ ,
- closed in  $(\mathbb{R} \times [1,2], \|\cdot\|_1)$ ,
- not closed in  $(\mathbb{R}^2, \|\cdot\|)$ ,

for  $\|\cdot\|$  the usual norm on  $\mathbb{R}^2$ .

- ex: •  $c_0$  is not closed in any  $(l^p, \| \cdot \|_p)$ ,  $1 \leq p < \infty$ :

Indeed, fix  $p \in [1, \infty)$ . It is enough to show that  $\exists x \in l^p \setminus c_0$  and  $(x_n)_{n \in \mathbb{N}}$  in  $c_0$  with  $x_n \xrightarrow{\| \cdot \|_p} x$ . In fact, we will show something stronger:

for all  $x \in l^p \setminus c_0$ ,  $\exists (x_n)_{n \in \mathbb{N}}$  in  $c_0$  with  $x_n \xrightarrow{\| \cdot \|_p} x$ .

Indeed, let  $x \in l^p \setminus c_0$ ,  $x = (y_1, \dots, y_n, \dots)$ .

for all  $n \in \mathbb{N}$ , let  $x_n = (y_1, \dots, y_n, 0, 0, \dots)$ . Then,  $x_n \xrightarrow[n \rightarrow \infty]{\| \cdot \|_p} x$ . Indeed:

$$\text{then } x_n - x = (\underbrace{0, \dots, 0}_{n \text{ terms}}, y_{n+1}, y_{n+2}, \dots), \text{ so } \|x_n - x\|_p = \left( \sum_{k=n+1}^{+\infty} |y_k|^p \right)^{1/p} \xrightarrow{n \rightarrow \infty} 0,$$

since  $\sum_{k=1}^{+\infty} |y_k|^p < \infty$  (as  $x \in l^p$ ).

- $c_0$  is not closed in  $(c_0, \| \cdot \|_\infty)$ , nor in  $(l^\infty, \| \cdot \|_\infty)$ . Can you show it?

Using the characterisation of closedness in a normed space via convergence of sequences, and due to the fact that any two norms in a finite dimensional vector space are equivalent, the following can be shown.

Thm 2.1: Let  $X$  be a vector space. Let  $Y$  be a finite dimensional subspace of  $X$ . Then,  $Y$  is closed in  $(X, \|\cdot\|)$ , for any norm  $\|\cdot\|$  on  $X$ .



This theorem is particularly interesting, because it shows there is a connection between the dimension of a vector space (which is a purely algebraic property, i.e. has nothing to do with metrics), and the closedness properties of the vector space inside larger spaces w.r.t. a (in fact any) norm.

We will soon see such a phenomenon occurring with regard to completeness of normed spaces (in particular, that a vector space with countably infinite dimension cannot be complete, i.e. Banach, w.r.t. any norm).

## Lecture 3 (7/10/2014).

### Banach spaces:

A normed vector space is Banach if it is a complete metric space with respect to the metric induced by the norm. In detail:

Def: Let  $(X, d)$  be a metric space. A sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  is Cauchy w.r.t.  $d$  if,  $\forall \varepsilon > 0$ ,  $\exists n_\varepsilon \in \mathbb{N}$ :  $\forall n, m \geq n_\varepsilon$ ,  $d(x_n, x_m) < \varepsilon$ .

Let  $(X, \| \cdot \|)$  be a normed space. A sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  is Cauchy w.r.t.  $\| \cdot \|$  if  $(x_n)_{n \in \mathbb{N}}$  is Cauchy w.r.t.  $d_{\| \cdot \|}$ , i.e.  
 $\forall \varepsilon > 0$ ,  $\exists n_\varepsilon \in \mathbb{N}$ :  $\forall n, m \geq n_\varepsilon$ ,  $\|x_n - x_m\| < \varepsilon$ .



Any sequence in a metric (or normed) space  $(X, d)$  (or  $(X, \| \cdot \|)$ ) that converges w.r.t.  $d$  (or  $\| \cdot \|$ ) is Cauchy w.r.t.  $d$  (or  $\| \cdot \|$ ). Note that this is true even

if the sequence converges to some point outside  $X$ .

for example, if  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $\ell^\infty$ , s.t.  $x_n \xrightarrow{\|\cdot\|_1} x \in \ell^1 \setminus \ell^\infty$ , then  $(x_n)_{n \in \mathbb{N}}$  is Cauchy w.r.t.  $\|\cdot\|_1$ .

Def.: Let  $(X, d)$  be a metric space.  $(X, d)$  is complete if every sequence in  $X$  that is Cauchy w.r.t.  $d$  converges in  $X$  w.r.t.  $d$ .



Seemingly, to check that a metric space is complete, we need to pick an arbitrary Cauchy sequence in the metric space, and show 2 things:

1. that it converges w.r.t. the metric, and
2. that the limit belongs to the metric space.

However, it can be shown that, for every metric space  $(X, d)$ , there exists some larger space  $X'$ , and an extension  $d'$  of  $d$  on  $X'$ , s.t.  $(X', d')$  is complete.

↑  
the completion of  $(X, d)$ .

Thus,  $(X', d')$  contains the limits of all Cauchy sequences in  $(X, d)$ .  
In other words, a Cauchy sequence in a metric space always converges w.r.t.  
the metric, in some larger space.

So, 1. above is not something that needs to be checked; it is always true.

This means that the important part of the definition of a complete metric  
space is that the limits of all Cauchy sequences lie in the space (and not  
that they exist: they always exist).

Def.: A normed space  $(X, \|\cdot\|)$  is **Banach** if  $(X, d_{\|\cdot\|})$  is complete, i.e. if, for every  
 $(x_n)_{n \in \mathbb{N}}$  in  $X$  that is Cauchy w.r.t.  $\|\cdot\|$ , there exists  $x \in X$  s.t.  $\|x_n - x\| \xrightarrow{n \rightarrow \infty} 0$ .

- ex.:
- $(L^p, \|\cdot\|_p)$ ,  $1 \leq p \leq \infty$  are all Banach spaces.
  - $(l^p, \|\cdot\|_p)$ ,  $1 \leq p \leq \infty$  are all Banach spaces.
  - $(C([a, b]), \|\cdot\|_\infty)$  is Banach,  $(C([a, b]), \|\cdot\|_1)$  is not. Can you show it?

- $(c_\infty, \|\cdot\|_p)$  is not Banach, for any  $1 \leq p \leq \infty$ : Indeed, let  $p \in [1, \infty]$ . for this  $p$ , pick  $x \in l^p \setminus c_\infty$  (there exists such an  $x$ , for all  $1 \leq p \leq \infty$ ).

There exists  $(x_n)_{n \in \mathbb{N}}$  in  $c_\infty$ , s.t.  $x_n \xrightarrow[n \rightarrow \infty]{\|\cdot\|_p} x$  (note that this is true for  $p = \infty$  only if  $x \in c_0$ , i.e. if  $x$  is a sequence converging to 0!).

Since  $(x_n)_{n \in \mathbb{N}}$  is convergent w.r.t.  $\|\cdot\|_p$ , it is Cauchy w.r.t.  $\|\cdot\|_p$ .

So,  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $(c_\infty, \|\cdot\|_p)$  that is Cauchy w.r.t.  $\|\cdot\|_p$  but does not converge in  $c_\infty$ .

Thus,  $(l^p, \|\cdot\|_p)$  is not Banach.

Thm 3.1.: Let  $(X, \|\cdot\|)$  be a Banach space. Let  $Y$  be a subspace of  $X$ . Then,  
 $(Y, \|\cdot\|)$  is Banach  $\iff Y$  is closed in  $(X, \|\cdot\|)$ .

Thm 3.2: Let  $X$  be a finite dimensional vector space. Then,  $(X, \|\cdot\|)$  is Banach, for any norm  $\|\cdot\|$  on  $X$ .



This last theorem connects the dimension of a vector space, which is a purely algebraic property, with its "ability" to be a Banach space w.r.t. some norm, something that, at first glance, has little to do with algebra.

We will see this phenomenon occurring again in this lecture.

### Baire's theorem :

Baire's theorem holds in complete metric spaces, and has numerous exciting applications. Here, we will see a weak version of the theorem, that is still very powerful.

Baire's thm: Let  $(X, d)$  be a complete metric space. Suppose that  
(weak version)

$X = \bigcup_{n=1}^{+\infty} f_n$ , where, for all  $n \in \mathbb{N}$ ,  $f_n$  is a closed subset of  $(X, d)$ .

Then,  $\exists n_0 \in \mathbb{N}: f_{n_0}^\circ \neq \emptyset$ , i.e.  $f_{n_0} \supseteq \underbrace{B_d(x, r)}$ , for some  $x \in X$  and some  $r > 0$ .

$\downarrow$   
an open ball  
of  $X$  w.r.t.  $d$

Baire's theorem has many applications. We demonstrate some.

App. 1:  $\mathbb{R}$  is an uncountable set:

Indeed, suppose that  $\mathbb{R}$  is countable, i.e. that  $\mathbb{R} = \{x_1, x_2, \dots, x_n, \dots\}$ .

Then,

$$\mathbb{R} = \bigcup_{n=1}^{+\infty} \underbrace{\{x_n\}}$$

complete w.r.t.  
usual metric

closed in  $\mathbb{R}$  w.r.t. usual metric.

By Baire's theorem,  $\exists n_0 \in \mathbb{N}: \{x_{n_0}\}$  contains some open ball of  $\mathbb{R}$ . This is a contradiction,  
w.r.t. the  
usual metric

because any open ball  $B(x, r)$  of  $\mathbb{R}$  (w.r.t. the usual metric) contains more than one element (for example,  $x$  and  $x + \frac{r}{2}$ ).

So,  $\mathbb{R}$  is uncountable.

The next application of Baire's theorem has to do with Banach spaces. Note that Banach spaces are complete w.r.t. the metric induced by the norm, so it makes sense to use Baire's theorem in them. Note that, in this setting, Baire's theorem says:

Let  $(X, \|\cdot\|)$  be a Banach space. Suppose that

$$X = \bigcup_{n=1}^{+\infty} F_n, \text{ where, } \forall n \in \mathbb{N}, F_n \text{ is a closed subset of } (X, \|\cdot\|).$$

Then,  $\exists n_0 \in \mathbb{N}: F_{n_0} \supseteq B(x, r)$ , for some  $x \in X$  and some  $r > 0$ .

In particular, application 2 will be based on the following.

Corollary of Baire's thm: Let  $(X, \|\cdot\|)$  be a Banach space. Suppose that  $X = \bigcup_{n=1}^{+\infty} F_n$ , where,  $\forall n \in \mathbb{N}$ ,  $F_n$  is a closed subspace of  $(X, \|\cdot\|)$ . Then,  $\exists n_0 \in \mathbb{N}: F_{n_0} = X$ .

Proof: By Baire's theorem,  $\exists n_0 \in \mathbb{N}: F_{n_0}$  contains some open ball of  $X$ . Since  $F_{n_0}$  is a subspace of  $X$ , this implies that  $F_{n_0} = X$ .

App. 2: A Banach space cannot have countably infinite dimension:

Let  $(X, \|\cdot\|)$  be a Banach space, with  $\dim X$  countably infinite. Then, there exist vectors  $e_1, e_2, \dots, e_n, \dots$  in  $X$ , s.t.

$$X = \text{span}\{e_1, e_2, \dots, e_n, \dots\}.$$

Thus,  $X = \bigcup_{n=1}^{+\infty} \underbrace{\text{span}\{e_1, e_2, \dots, e_n\}}_{\text{finite dimensional, so closed in } (X, \|\cdot\|)}.$

Banach w.r.t.  $\|\cdot\|$

Therefore,  $\exists n_0 \in \mathbb{N}$ :  $\text{span}\{e_1, \dots, e_{n_0}\}$  contains an open ball of  $X$ . Since  $\text{span}\{e_1, \dots, e_{n_0}\}$  is a subspace of  $X$ , this implies that  $\text{span}\{e_1, \dots, e_{n_0}\} = X$ .

Thus,  $X$  is finite dimensional, contradiction.

So,  $\dim X$  cannot be countably infinite.

The above can be rephrased as follows:

Let  $X$  be a vector space with countably infinite dimension. Then,  $(X, \|\cdot\|)$  is not Banach, for any norm  $\|\cdot\|$  on  $X$ .

ex:  $(\ell^\infty, \|\cdot\|)$  is not Banach for any norm  $\|\cdot\|$

(and not just for  $\|\cdot\|_p$ ,  $1 \leq p \leq +\infty$ )

:

Indeed, the reason is that  $\dim \ell^\infty$  is countably infinite, because  $\ell^\infty$  has a countably infinite basis: the set

$$\{e_1, e_2, \dots, e_n, \dots\},$$

where, for each  $i \in \mathbb{N}$ ,  $e_i = (0, 0, \dots, 0, \underset{\uparrow}{1}, 0, 0, \dots) \in \ell^\infty$ .

$i$ -th term

To sum up, there exists the following relationship between the dimension of a vector space  $X$  and its "ability" to be a Banach space:

- If  $\dim X < +\infty$ , then  $(X, \|\cdot\|)$  Banach, for any norm  $\|\cdot\|$  on  $X$ .
- If  $\dim X$  is countably infinite, then  $(X, \|\cdot\|)$  not Banach, for any norm  $\|\cdot\|$  on  $X$ .
- If  $\dim X$  is uncountably infinite, then  $(X, \|\cdot\|)$  may be Banach for some

norms  $\|\cdot\|$  on  $X$ , and may not be Banach for others.

for instance,  $(C([a,b]), \|\cdot\|_\infty)$  is Banach,  $(C([a,b]), \|\cdot\|_1)$  is not Banach  
(note that this immediately implies that  $\dim C([a,b])$  is uncountably infinite).

In future lectures, we will see more applications of Baire's theorem:

App. 3: The Uniform Boundedness Principle.

App. 4: The Open Mapping Theorem and the Closed Graph Theorem.

## Lecture 4 (9/10/2014)

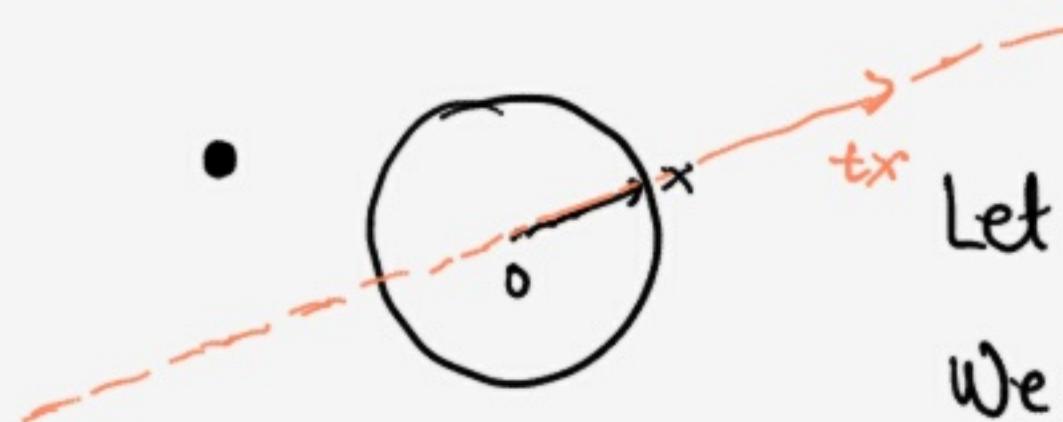
### functional Analysis:

Notation: Let  $(X, \|\cdot\|)$  be a normed space. We denote:

$$S_X(x, r) := \{y \in X : \|y - x\| = r\},$$

$$\bar{B}_X(x, r) := \{y \in X : \|y - x\| \leq r\}.$$

Observations: • Let  $x \in (X, \|\cdot\|)$ . Then,  $\frac{x}{\|x\|} \in S_X(0, 1)$ . Therefore  $\frac{x}{\|x\|}$  is a unit vector, with the same direction as  $x$  (i.e., linearly dependent with  $x$ ).



Let  $f: (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$  be a linear map.

We want to understand how  $f$  changes under dilations, i.e.

how  $\|f(tx)\|_y$  changes as  $t$  varies in  $\mathbb{R}$ :  $\|f(tx)\|_y = \|t \cdot f(x)\|_y = |t| \cdot \|f(x)\|_y$ , Ht. So:

- $\|f(tx)\|_y$  runs from 0 to  $+\infty$  as  $|t|$  runs from 0 to  $+\infty$ . (as long as  $f(x) \neq 0$  !)  
In fact,  $\|f(tx)\|$  is strictly increasing in  $|t|$ .
- When  $t \nearrow 1$ ,  $tx \rightarrow x$  and  $\|f(tx)\|_y \nearrow \|f(x)\|_y$ . [In particular, any  $y \in Y$  on line segment connecting 0 with  $x$  has  $\|f(y)\|_y \leq \|f(x)\|_y$  ( $\leq$  if  $f(x) \neq 0$ ).]

The above imply that:

$$\boxed{\begin{aligned} \bullet \quad \sup_{x \in S_X(0,1)} \|f(x)\|_y &= \sup_{x \in B_X(0,1)} \|f(x)\|_y = \sup_{x \in B_X(0,1)} \|f(x)\|_y \\ \bullet \quad \forall M > 0, \quad \sup_{x \in S_X(0,M)} \|f(x)\|_y &= M \cdot \sup_{x \in S_X(0,1)} \|f(x)\|_y \end{aligned}}$$

It turns out that  $f$  is continuous iff these supremums are finite,

$$\text{i.e. iff } \sup_{x \in S_x(0,1)} \|f(x)\|_Y < +\infty.$$

Let us see this in more detail. We start with a definition that, at first glance, may seem irrelevant to the considerations above (but it is not).

Def: Let  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  be normed spaces. A linear map  $f: X \rightarrow Y$  is **bounded** if  $\exists M < +\infty$ , such that

$$\|f(x)\|_Y \leq M \cdot \|x\|_X, \quad \forall x \in X. \quad (*)$$



Note that a linear map  $f: X \rightarrow Y$  that is not identically 0 cannot be bounded in the classical sense, i.e.  $\nexists M < +\infty$  s.t.  $\|f(x)\|_Y \leq M \quad \forall x \in X$ . Indeed, as we have already mentioned, if  $f(x) \neq 0$ , then the line connecting  $x$  with 0 is an unbounded (w.r.t.  $\|\cdot\|_Y$ ) subset of  $Y$ .

Def: Let  $f: (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$  be a bounded linear map. We define

$$\|f\| := \inf \{M : \textcircled{*} \text{ holds for } f\}.$$



- Note that, since  $f$  is bounded,  $\exists M < \infty$  s.t.  $\textcircled{*}$  holds for  $f$ , so the infimum above is finite, i.e.  $\|f\| < \infty$ .
- $\boxed{\|f(x)\|_Y \leq \|f\| \cdot \|x\|_X}$ . Indeed, if  $(M_n)_{n \in \mathbb{N}}$  is a sequence in  $\{M : \textcircled{*} \text{ holds for } f\}$  that converges to the infimum of the set, i.e.  $M_n \rightarrow \|f\|$ , then,  $\forall x \in X$ ,

$$\|f(x)\|_Y \leq M_n \cdot \|x\|_X \text{ for all } n \in \mathbb{N}, \text{ thus}$$

$$\underbrace{\lim_{n \rightarrow \infty} \|f(x)\|_Y}_{\|f(x)\|_Y} \leq \lim_{n \rightarrow \infty} (M_n \cdot \|x\|_X) = \left( \lim_{n \rightarrow \infty} M_n \right) \cdot \|x\|_X = \|f\| \cdot \|x\|_X.$$

$$\|f(x)\|_Y$$

In particular, this means that  $\|f\| \in \{M : \textcircled{*} \text{ holds for } f\}$ , and thus the infimum above is a minimum.

Let  $B(X, Y) := \{f: X \rightarrow Y : f \text{ linear and bounded}\} \rightsquigarrow$  a vector space.

The function  $\|\cdot\|: B(X, Y) \rightarrow \mathbb{R}$  that we just defined happens to be a norm on  $B(X, Y)$ .

We call it the operator norm on  $B(X, Y)$ .

Now, we notice that there exists a connection between the boundedness of a linear map  $f: X \rightarrow Y$  and whether  $f(B_X(0,1))$  is bounded in  $Y$ .

(also,  
 $f(S_X(0,1)),$   
 $f(\bar{B}_X(0,1))$ )

Indeed, let  $f: (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$  be a linear map. Let  $M < \infty$ . Then,

$$\begin{aligned} \|f(x)\|_Y \leq M \cdot \|x\|_X \text{ for all } x \in X &\iff \|f\left(\frac{x}{\|x\|_X}\right)\|_Y \leq M, \text{ for all } x \in X \setminus \{0\} \\ &\iff \|f(x)\|_Y \leq M, \text{ for all } x \in S_X(0,1) \\ &\iff \sup_{x \in S_X(0,1)} \|f(x)\|_Y \leq M, \text{ i.e. } f(\bar{B}_X(0,1)) \subseteq B_Y(0, M). \end{aligned}$$

This implies that

$$\sup_{x \in S_X(0,1)} \|f(x)\|_Y = \inf\{M : \textcircled{*} \text{ holds for } f\}.$$

Therefore: •  $\sup_{x \in S_X(0,1)} \|f(x)\|_Y < +\infty \iff \inf\{M : \textcircled{*} \text{ holds for } f\} < +\infty \iff f \text{ bounded.}$

In other words,  $f$  bounded  $\iff f(S_X(0,1))$  is bounded in  $Y$  (w.r.t.  $\|\cdot\|_Y$ )

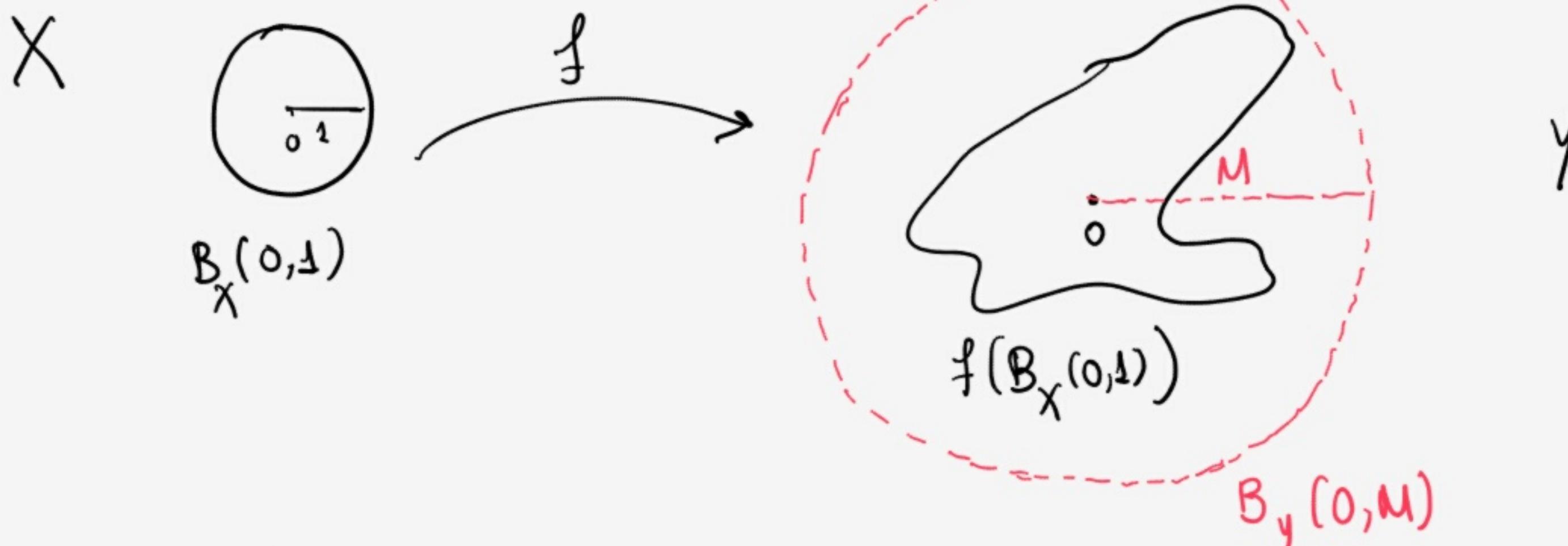
(and also  
 $f(B_X(0,1)),$   
 $f(\bar{B}_X(0,1))$ )

• If  $f$  is bounded, then  $\sup_{x \in S_X(0,1)} \|f(x)\| = \|f\|.$



This is a good way to understand when a linear map  $f: X \rightarrow Y$  is bounded, and, if yes, what its norm is.

In particular, to see if  $f$  is bounded, we need to look at  $f(B_X(0,1))$ .



$f$  is bounded  $\Leftrightarrow f(B_X(0,1))$  is bounded, i.e.  $\exists M < \infty : f(B_X(0,1)) \subseteq B_Y(0,M)$ .

If there exists such an  $M < \infty$ , then  $\|f\| \leq M$ , and  $\|f\|$  is the radius of the smallest ball in  $Y$ , centered at 0, that contains  $f(B_X(0,1))$  (and  $f(\bar{B}_X(0,1))$ , and  $f(S_X(0,1))$ ).

Thm 4.1: Let  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  be normed spaces. Then,

$(B(X, Y), \|\cdot\|)$  is Banach  $\Leftrightarrow (Y, \|\cdot\|_Y)$  is Banach.

In particular, the space  $X^* := B(X, \mathbb{R})$ , the space of bounded linear functionals [i.e. of bounded linear maps from  $(X, \|\cdot\|_X)$  to  $(\mathbb{R}, |\cdot|)$ ] is Banach with the operator norm.

Thm 4.2: Let  $f: (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$  be a linear map. Then, the following are equivalent:

- (i)  $f$  continuous
- (ii)  $f$  continuous at 0.
- (iii)  $f$  bounded.

Proof (to be completed in next lecture):

(i)  $\Rightarrow$  (ii) : obvious.

(iii)  $\Rightarrow$  (i) :  $f$  bounded  $\Rightarrow f$  Lipschitz continuous, so  $f$  continuous.

Indeed, since  $f$  is bounded,  $\exists M < +\infty$ :  $\|f(x)\|_y \leq M \cdot \|x\|_X, \forall x \in X$ .

In particular,  $\|f(x-y)\|_y \leq M \cdot \|x-y\|_X, \forall x, y \in X$ .

But  $f(x-y) = f(x) - f(y)$ , since  $f$  linear.

So,  $\|f(x) - f(y)\|_y \leq M \cdot \|x-y\|_X, \forall x, y \in X$ , so  $f$  Lipschitz continuous  
 $\Rightarrow f$  continuous.

(Let us prove why  $f$  continuous in detail: Let  $x_0 \in X$ .  $f$  continuous at  $x_0$ :

It suffices to show that, for every  $(x_n)_{n \in \mathbb{N}}$  in  $X$  with  $x_n \xrightarrow{\|\cdot\|_X} x_0$ ,

it holds that  $f(x_n) \xrightarrow{\|\cdot\|_y} f(x_0)$ . Indeed:  $\|f(x_n) - f(x_0)\|_y \leq M \cdot \|x_n - x_0\|_X \rightarrow 0$ ,

so  $f(x_n) \xrightarrow{\|\cdot\|_y} f(x_0)$ ).

## Lecture 5 (14/10/2014)

Thm 4.2: Let  $f: (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$  be a linear map. Then, the following are equivalent:

- (i)  $f$  continuous
- (ii)  $f$  continuous at 0.
- (iii)  $f$  bounded.

Proof (continued):

(ii)  $\Rightarrow$  (iii):  $f$  continuous at 0, i.e.:  $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$  s.t.  $\left. \begin{array}{l} \text{if } \|x-0\|_X < \delta_\varepsilon, \\ \text{then } \|f(x)-f(0)\|_Y < \varepsilon \end{array} \right\}$ .

In particular (for  $\varepsilon=1$ ):

$\exists \delta > 0$  s.t.: (if  $\|x\|_X < \delta$ , then  $\|f(x)\|_Y < 1$ ). This means that

$f(B_X(0, \delta)) \subseteq B_Y(0, 1)$ , thus the image of some ball centered at 0 is bounded in Y. Therefore,  $f(B_X(0, 1))$  is bounded in Y.

$$B_X(0, 1) = \frac{1}{\delta} \cdot B_X(0, \delta), \text{ so}$$

$$f(B_X(0, 1)) = \frac{1}{\delta} \cdot f(B_X(0, \delta)) \subseteq \frac{1}{\delta} \cdot B_Y(0, 1) = B_Y(0, \frac{1}{\delta}).$$



due to the  
properties of  
a norm

Thus, f is bounded.

Exercise : Let  $f_n: \mathbb{C}^\infty \rightarrow \mathbb{R}$ ,

$$x = (x_1, \dots, x_n, 0, 0, \dots) \mapsto \sum_{i=1}^n x_i. \quad f_n \text{ is a linear map.}$$

Show that  $f_{n_0} : (\mathbb{C}_\infty, \|\cdot\|_1) \rightarrow \mathbb{R}$  is bounded with norm 1,

and  $f_{n_0} : (\mathbb{C}_\infty, \|\cdot\|_\infty) \rightarrow \mathbb{R}$  is bounded with norm  $n_0$ .

Solution: • Consider  $f_{n_0} : (\mathbb{C}_\infty, \|\cdot\|_1) \rightarrow \mathbb{R}$ .

for all  $x \in \mathbb{C}_\infty$ ,  $|f_{n_0}(x)| = \left| \sum_{i=1}^{n_0} x_i \right| \leq \sum_{i=1}^{n_0} |x_i| \leq \sum_{i=1}^{+\infty} |x_i| = \|x\|_1$ , so

$$|f_{n_0}(x)| \leq \|x\|_1.$$

Therefore,  $f_{n_0}$  bounded and  $\|f_{n_0}\| \leq 1$ .

In fact,  $\|f_{n_0}\| = 1$ : for  $x = (1, 0, 0, \dots)$ , we have that  $x \in \mathbb{C}_\infty$ ,  $\|x\|_1 = 1$  and

$$|f_{n_0}(x)| = |1| = 1, \text{ so } \sup_{\substack{x \in S \\ ((\mathbb{C}_\infty, \|\cdot\|_1))}} |f_{n_0}(x)| \geq |f_{n_0}(x_0)| = 1.$$

$\underbrace{\hspace{10em}}$   
 $\parallel$   
 $\|f_{n_0}\|$

- Consider  $f_{n_0} : (\mathbb{C}_\infty, \|\cdot\|_\infty) \rightarrow \mathbb{R}$ .

for all  $x \in \mathbb{C}_\infty$ ,  $|f_{n_0}(x)| = \left| \sum_{i=1}^{n_0} x_i \right| \leq \sum_{i=1}^{n_0} |x_i| \leq \underbrace{\sum_{i=1}^{n_0} \|x\|_\infty}_{\|x\|_\infty} = n_0 \cdot \|x\|_\infty$ , so

$$|x_i| \leq \|x\|_\infty, \forall i$$

$$|f_{n_0}(x)| \leq n_0 \cdot \|x\|.$$

Therefore,  $f_{n_0}$  is bounded and  $\|f_{n_0}\| \leq n_0$ .

In fact,  $\|f_{n_0}\| = n_0$ : for  $x_0 = (\underbrace{1, 1, \dots, 1}_{n_0 \text{ (or more) terms}}, 0, \dots)$ , we have that  $x_0 \in \mathbb{C}_\infty$ ,  $\|x_0\|_\infty = 1$ , and

$$|f_{n_0}(x_0)| = \left| \sum_{i=1}^{n_0} 1 \right| = n_0, \quad \text{so} \quad \sup_{\substack{x \in S \\ (\mathbb{C}_\infty, \|\cdot\|_1)}} |f_{n_0}(x)| \geq |f_{n_0}(x_0)| = n_0.$$



Note that this demonstrates how dramatically the norm of a linear map can

change when the norm on the domain or the image changes.

→ Let  $(X, \|\cdot\|)$  be a normed space. We denote by  $X^{**}$  the space  $(X^*)^*$ , the dual of the dual of  $X$ .  $X^{**}$  is the space of bounded linear maps  $f: X^* \rightarrow \mathbb{R}$

  
this is the space of bounded linear maps  $g: (X, \|\cdot\|) \rightarrow \mathbb{R}$ , equipped with the operator norm.

Thm 5.1: Let  $(X, \|\cdot\|_X)$  be a normed space. There exists an isometric inclusion  $?$   
of  $(X, \|\cdot\|_X)$  inside  $(X^{**}, \|\cdot\|)$ :

↳ the operator norm  
(the norm a linear map  $f: (X^*, \|\cdot\|_{X^*}) \rightarrow \mathbb{R}$ )

$$\hat{\cdot}: (X, \|\cdot\|_X) \rightarrow (X^{**}, \|\cdot\|)$$

$$x \quad \rightarrow \quad \hat{x}: X^* \rightarrow \mathbb{R}$$

$$f \quad \rightarrow \quad f(x).$$

We denote :  $X \overset{\hat{\cdot}}{\hookrightarrow} X^{**}$ . (Note that  $\hat{\cdot}: X \rightarrow X^{**}$  is linear).

Proof: We need to show that,  $\forall x \in X$ ,  $\hat{x}$  is a bounded linear map, with  $\|\hat{x}\| = \|x\|$ .

Let  $x \in X$ . Then,

$$\hat{x}: X^* \rightarrow \mathbb{R}$$

$$f \rightarrow f(x).$$

- $\hat{x}$  is a linear map:  $\hat{x}(f_1 + f_2)(x) = (f_1 + f_2)(x) = f_1(x) + f_2(x) = \hat{x}(f_1) + \hat{x}(f_2)$ ,  $\forall f_1, f_2 \in X^*$ .
- $\hat{x}(\lambda f) = (\lambda f)(x) = \lambda \cdot f(x)$ , for all scalars  $\lambda$ ,  $\forall f \in X^*$ .
- $\|\hat{x}\| = \|x\|$ : We need to find the norm of  $\hat{x}: X^* \rightarrow \mathbb{R}$ .

We have that,  $\forall f \in X^*$ :

$$|\hat{x}(f)| = |f(x)| \leq \|f\| \cdot \|x\|_X = \|x\|_X \cdot \|f\|, \text{ so}$$

$$f \in X^*,$$

i.e. bounded

$$|\hat{x}(f)| \leq \underbrace{\|x\|}_X \cdot \|f\|, \quad \forall f \in X^*. \quad \text{Therefore, } \hat{x} : X^* \rightarrow \mathbb{R} \text{ bounded, and } \|\hat{x}\| \leq \|x\|.$$

↓  
constant independent  
of  $f$ .

In fact,  $\|\hat{x}\| = \|x\|$ :  $\exists f \in X^*$ , with  $\|f\|=1$ , s.t.  $\frac{f(x)}{\|\hat{x}(f)\|} = \|x\|_X$  (by the Hahn-Banach theorem),

$$\text{thus } \sup_{\substack{f \in S(0,1) \\ X^*}} |\hat{x}(f)| \geq \|x\|_X.$$

$$\underbrace{\begin{array}{c} f \in S(0,1) \\ X^* \end{array}}_{\|\hat{x}\|}$$



Note that, since  $\|\hat{x}\| = \|x\|_X$  for all  $x \in X$ ,  $\hat{\cdot}: X \rightarrow X^{**}$  is a bounded linear map with norm 1.



Let us explain how the Hahn-Banach theorem implies that, for  $x \in (X, \|\cdot\|_X)$ , there exists  $f \in X^*$ , with  $\|f\|=1$ , such that  $f(x)=\|x\|$ .

An important Corollary of the H-B theorem is the following:

Thm: Let  $(X, \|\cdot\|)$  be a normed space, and  $Y$  a subspace of  $X$ .

Let  $f: (Y, \|\cdot\|) \rightarrow \mathbb{R}$  be a bounded linear map.

Then, there exists  $\tilde{f}: (X, \|\cdot\|) \rightarrow \mathbb{R}$ , with  $\tilde{f}|_Y = f$ ,

and  $\|\tilde{f}\| = \|f\|$ .

Now, back to our problem:

Let  $y = \text{span}\{x\}$  ( $\alpha$  1-dim subspace of  $X$ ).

We define a linear map  $g: \text{span}\{x\} \rightarrow \mathbb{R}$ , that does what we want: sends  $x$  to  $\|x\|$ :

$$x \rightarrow \|x\|$$

$$(\text{and } \lambda x \rightarrow \lambda \cdot \|x\|, \text{ th } \lambda \text{ scalars}).$$

Now,  $g$  is a bounded linear map, with norm 1:

$$\begin{aligned} \text{for each } y \in \text{span}\{x\}, \quad y &= \lambda x \text{ for some scalar } \lambda, \text{ so } |g(y)| = |g(\lambda x)| = |\lambda \cdot g(x)| = \\ &= |\lambda \cdot \|x\|| = \|\lambda x\| = \|y\|, \end{aligned}$$

so  $|g(y)| = \|y\|, \forall y \in \text{span}\{x\}$ , so  $g$  is bounded with norm 1.

By the H-B theorem,  $g$  can be extended to a linear map  $\tilde{g}: (X, \|\cdot\|) \rightarrow \mathbb{R}$ , that is bounded and has the same norm as  $g$ .

Therefore,  $\|\tilde{g}\| = 1$  and  $\tilde{g}(x) = g(x) = \|x\|$ .

## Uniform Boundedness Principle:

Let  $(X, \|\cdot\|_X)$  be a Banach space,  $(Y, \|\cdot\|_Y)$  a normed space.

Let  $\mathcal{T}$  be a collection of bounded linear maps  $T: X \rightarrow Y$ .

If,  $\forall x \in X$ ,  $\{T(x) : T \in \mathcal{T}\}$  is bounded in  $Y$ ,

$$\left( \begin{array}{l} \text{i.e. } \sup_{T \in \mathcal{T}} \|T(x)\|_Y < +\infty, \\ \text{i.e. } \exists M_x < +\infty, \text{ s.t. } \|T(x)\|_Y \leq M_x \text{ for all } T \in \mathcal{T} \end{array} \right)$$

then  $\mathcal{T}$  is bounded in  $B(X, Y)$ .

$$\left( \begin{array}{l} \text{i.e. } \sup_{T \in \mathcal{T}} \|T\| < +\infty, \\ \text{i.e. } \exists M < +\infty \text{ s.t. } \|T\| \leq M, \text{ for all } T \in \mathcal{T}, \\ \text{i.e. the maps in } \mathcal{T} \text{ are uniformly bounded} \end{array} \right)$$

Proof: The proof is done using Baire's theorem on the Banach space  $(X, \|\cdot\|_X)$ .

(to be continued) In particular,

$$X = \bigcup_{n=1}^{+\infty} \{x \in X : \sup_{T \in T} \|T(x)\|_Y \leq n \cdot \|x\|_X \text{ for all } T \in T\}$$

$f_n$

(these sets will turn out to be closed subsets of  $X$ , so by Baire's thm one will have non-empty interior.

from that we will deduce what we want).

Indeed, let  $x \in X$ . By our assumption,

$$\sup_{T \in T} \|T(x)\|_Y < +\infty, \text{ so } \exists n \in \mathbb{N} \text{ s.t.}$$

fixed number

$$\sup_{T \in T} \|T(x)\|_Y \leq n \cdot \underbrace{\|x\|_X}_{\text{fixed number}} \quad (\text{for } \|x\|_X \neq 0, n \cdot \|x\|_X \rightarrow +\infty \text{ as } n \rightarrow +\infty, \text{ while for } \|x\|_X = 0, x = 0, \text{ so } \sup_{T \in T} \|T(x)\|_Y = 0).$$

So,  $x \in f_n$ .

## Lecture 6 (16/10/2014).

Proof of UBP: (slight modification to the definition of the sets  $f_n$  from previous lecture, to make the proof a little less technical):

We know that,  $\forall x \in X$ ,  $\{T(x) : T \in \Gamma\}$  is bounded in  $Y$ , i.e.

$$\sup \{\|T(x)\|_Y : T \in \Gamma\} < +\infty.$$

Therefore,  $X = \bigcup_{n=1}^{+\infty} \underbrace{\{x \in X : \sup \{\|T(x)\|_Y : T \in \Gamma\} \leq n\}}_{\|f_n\|}$ .

Indeed, if  $x \in X$ , then  $\sup \{\|T(x)\|_Y : T \in \Gamma\} < +\infty$ , so  $\exists n \in \mathbb{N}$  s.t.

$$\sup \{\|T(x)\|_Y : T \in \Gamma\} \leq n.$$

(note that this  $n$  depends on  $x$ ).

Now: •  $(X, \|\cdot\|_X)$  is a Banach space.

- for all  $n \in \mathbb{N}$ ,  $f_n$  closed subset of  $(X, \|\cdot\|_X)$ :

Indeed, let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $f_n$ , s.t.  $x_n \xrightarrow[\|\cdot\|_X]{} x$ , for some  $x \in X$ .

We will show that  $x \in f_n$  (this proves that  $f_n$  is closed):

Let  $T \in T$ . For all  $n \in \mathbb{N}$ ,  $x_n \in f_n \Rightarrow \|T(x_n)\|_y \leq n$ .

Also,  $T$  is continuous, so  $T(x_n) \xrightarrow[\|\cdot\|_y]{} T(x)$ , thus  $\|T(x_n)\|_y \xrightarrow{} \|T(x)\|_y$   $\Rightarrow$

$\Rightarrow \|T(x)\|_y \leq n$ . Since  $T$  was an arbitrary element of  $T$ , it follows that

$\|T(x)\|_y \leq n$  for all  $T \in T$ , i.e.  $\sup \{\|T(x)\|_y : T \in T\} \leq n$ , so  $x \in f_n$ .

Therefore, by Baire's theorem,  $\exists n_0 \in \mathbb{N}$  s.t.  $f_{n_0}^o \neq \emptyset$ , i.e.

$f_{n_0} \supseteq B_X(x_0, r)$ , for some  $x_0 \in X$  and  $r > 0$ .

Thus, for all  $x' \in B_X(x_0, r)$ ,  $\sup\{\|T(x')\|_y : T \in \mathcal{T}\} \leq n_0$ ,

i.e.  $\|T(x')\|_y \leq n_0$ , for all  $T \in \mathcal{T}$ .

We want to show something similar for all points of  $B_X(0, 1)$ .

The idea is that, since we can get  $B_X(0, 1)$  by translating and dilating  $B_X(x_0, r)$ , we can understand the behaviour of the linear maps  $T$  on  $B_X(0, 1)$  as long as we understand it on  $B_X(x_0, r)$ .

Indeed,  $B_X(0, 1) = \frac{1}{r} \cdot B_X(0, r) = \frac{1}{r} \cdot (B_X(x_0, r) - x_0)$ . So:

Let  $x \in B_X(0, 1)$ . If  $x' \in B_X(x_0, r)$ , s.t.  $x = \frac{1}{r} \cdot (x' - x_0)$  (by the above).

Now, let  $T \in \mathcal{T}$ . We have that

$$\begin{aligned} \|T(x)\|_y &= \|T\left(\frac{1}{r}(x' - x_0)\right)\|_y = \frac{1}{r} \cdot \|T(x') - T(x_0)\|_y \leq \frac{1}{r} \cdot (\|T(x')\| + \|T(x_0)\|) \leq \\ &\leq \frac{2n_0}{r}, \text{ as } x', x_0 \in B_X(x_0, r). \end{aligned}$$

triangle  
inequality

$$\text{Thus, } \sup_{\substack{x \in B_X(0,1) \\ X}} \|T(x)\| \leq \frac{2n_0}{r} \Rightarrow \|T\| \leq \frac{2n_0}{r}$$

$\underbrace{\phantom{\sup_{\substack{x \in B_X(0,1)}} \|T(x)\|}}$   
 $\parallel$   
 $\|T\|$

a fixed number,  
independent of T.

Therefore,  $\sup \{\|T\| : T \in \mathcal{T}\} \leq \frac{2n_0}{r} (< \infty)$ , i.e.  $\mathcal{T}$  is bounded in  $B(X, Y)$   
 (i.e., the maps in  $\mathcal{T}$  are uniformly bounded).

The proof is complete.

App. 1: Let  $(X, \|\cdot\|_X)$  be a normed space. Let  $K \subseteq X$ .

Then,  $K$  bounded in  $X \iff f(K)$  is bounded in  $\mathbb{R}$ , for all  $f \in X^*$ .

Proof: ( $\Rightarrow$ )  $K$  bounded in  $X$ . Let  $f \in X^*$ .  $f$  is a bounded linear map, so it sends balls centered at 0 to bounded sets, and thus bounded sets to bounded sets:

Since  $K$  bounded,  $\exists M < \infty$  s.t.  $K \subseteq B_X(0, M) \Rightarrow f(K) \subseteq \underbrace{f(B_X(0, M))}_{\text{bounded, as it is } = M \cdot f(B_X(0, 1))}$

( $\Leftarrow$ ) The idea is to see each  $x \in K$  as a bounded linear map, with (operator) norm equal to  $\|x\|_X$ , and then uniformly bound these maps by UBP.

$\sup_{x \in B_X(0, 1)} \|f(x)\| < \infty$  since  $f$  bounded.

Indeed, for each  $x \in K$ , the map

$\hat{x} : X^* \rightarrow \mathbb{R}$  is linear and bounded, with  $\|\hat{x}\| = \|x\|_X$ .

It therefore suffices to bound  $\{\hat{x} : x \in K\}$  (because  $K$  bounded  $\Leftrightarrow \{x : x \in K\}$  bounded)

$$\Leftrightarrow \{\|x\| : x \in K\} \text{ bounded} \underset{\text{def}}{\Leftrightarrow} \|x\|_X = \|\hat{x}\| \forall x \in X$$

$$\Leftrightarrow \{\|\hat{x}\| : x \in K\} \text{ bounded, i.e. } \{\hat{x} : x \in K\} \text{ bounded in } B(X, Y)$$

It is enough to show that the family  $\{\hat{x}: x \in K\}$  of linear maps satisfies the conditions in the statement of the U.B.P..

Indeed: • Each  $\hat{x}$  is linear and bounded from  $X^*$  to  $\mathbb{R}$ , and  $X^*$  is Banach.

- For every  $f \in X^*$ ,  $\{\hat{x}(f): x \in K\}$  is bounded (in  $\mathbb{R}$ ):

$$\text{Let } f \in X^*. \text{ for each } x \in K, |\hat{x}(f)| = |f(x)| \leq \|f\| \cdot \|x\|_X,$$

$$\begin{aligned} \text{so } \sup\{|\hat{x}(f)| : x \in K\} &\leq \sup\{\|f\| \cdot \|x\|_X : x \in K\} = \\ &= \|f\| \cdot \sup\{\|x\|_X : x \in K\} < \infty, \text{ as } K \text{ is bounded in } X \\ &\quad (\text{i.e. } \sup\{\|x\|_X : x \in K\} < \infty). \end{aligned}$$

### $L^p$ -spaces:

Let  $(X, \mu)$  be a measure space. We will be interested in measurable maps  $f: X \longrightarrow \mathbb{C}$ , where  $\mathbb{C}$  (or  $\mathbb{R}$ ) is equipped with the Lebesgue measure on the Borel  $\sigma$ -algebra. (or  $\mathbb{TR}$ )

We first define a relation  $\sim$  on the space of measurable functions  $f: X \rightarrow \mathbb{C}$  :  
(or  $\mathbb{R}$ )

We say that  $f_1 \sim f_2$  if  $f_1, f_2$  differ only on a set of  $\mu$ -measure 0  
(i.e., if  $\mu(\{x \in X : f_1(x) \neq f_2(x)\}) = 0$ ,  
i.e. if  $f_1 = f_2 \mu$ -a.e.).  
an equivalence relation

Def. Let  $(X, \mu)$  be a measure space. For  $1 \leq p \leq +\infty$ , we define

$$L^p(X, \mu) := \overline{L^p(X, \mu)}_{\sim},$$

where, for  $1 \leq p \leq +\infty$ ,  $\overline{L^p}(X, \mu) := \left\{ f: X \xrightarrow[\text{(or } \mathbb{R}\text{)}]{} \mathbb{C} \text{ measurable, s.t. } \int_X |f|^p d\mu < +\infty \right\},$

while for  $p = +\infty$ ,  $\overline{L^\infty}(X, \mu) := \left\{ f: X \xrightarrow[\text{(or } \mathbb{R}\text{)}]{} \mathbb{C} \text{ measurable, s.t. } f \text{ is essentially bounded} \right\}.$

[Note that a measurable  $f: X \xrightarrow[\text{(or } \mathbb{R}\text{)}]{} \mathbb{C}$  is **essentially bounded** if  $\exists M < +\infty$  s.t.  $|f| \leq M$   $\mu$ -a.e.,]  
i.e.  $\mu(\{x \in X : |f| > M\}) = 0$

for  $1 \leq p \leq +\infty$ , we define  $\|\cdot\|_p : L^p(X, \mu) \rightarrow \mathbb{R}$ , s.t.  $\|f\|_p = \left( \int_X |f|^p d\mu \right)^{1/p}$ .

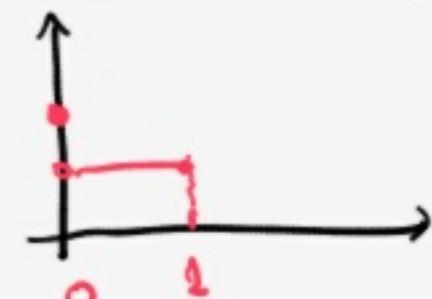
for  $p=+\infty$ , we define  $\|\cdot\|_\infty : L^\infty(X, \mu) \rightarrow \mathbb{R}$ , s.t.  $\|f\|_\infty = \text{essential supremum of } |f| = \inf \{M < +\infty : |f| \leq M \text{ } \mu\text{-a.e.}\}$ .

→ for each  $1 \leq p \leq +\infty$ ,  $\|\cdot\|_p$  is a norm on  $L^p(X, \mu)$ . In fact,  $(L^p(X, \mu), \|\cdot\|_p)$  is Banach.



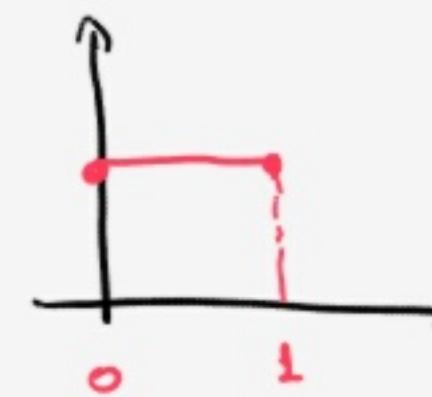
Because of the equivalence relation  $\sim$  in the definition of  $L^p(X, \mu)$ , whenever we consider a subspace of an  $L^p$  space, such as  $C([a, b])$  ( $\subseteq L^p(\mathbb{R}, \text{Lebesgue measure})$ ), and we assign the  $\|\cdot\|_p$  norm to it, we consider the functions in the subspace as different only when they are not equivalent.

for example, the function



is in  $(C([0, 1]), \|\cdot\|_1)$ , because in this

space it is equal to the function



, which is continuous.



$\ell^p = L^p(\mathbb{N}, \nu)$ , where  $\nu$  is the counting measure ( $\nu(E) = \#E$ ,  $\forall E \subseteq \mathbb{N}$ ),  
and the  $\ell^p$ -norm is the  $\|\cdot\|_p$  on  $L^p(\mathbb{N}, \nu)$

Indeed, let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{C}$  (or  $\mathbb{R}$ ). Then, we can see  $(x_n)_{n \in \mathbb{N}}$  as a  
function  $x: \mathbb{N} \rightarrow \mathbb{C}$  (or  $\mathbb{R}$ ), with  $x(n) = x_n$ ,  $\forall n \in \mathbb{N}$ .

$$\text{In particular, for } 1 \leq p < \infty: \int_{\mathbb{N}} |x|^p d\nu = \sum_{n \in \mathbb{N}} |x(n)|^p d\nu(n) = |x(1)|^p \nu(\{1\}) + |x(2)|^p \nu(\{2\}) + \dots = \\ = |x(1)|^p + |x(2)|^p + \dots = \\ = \sum_{n=1}^{+\infty} |x(n)|^p = \sum_{n=1}^{+\infty} |x_n|^p,$$

so  $\|(x_n)_{n \in \mathbb{N}}\|_p = \|x\|_p$   
 $\hookrightarrow$  the  $L^p(\mathbb{N}, \nu)$ -norm of  $x \in L^p(\mathbb{N}, \nu)$ ,

while for  $p=\infty$ : essential supremum of  $|x| = \sup|x| = \sup_{n \in \mathbb{N}} |x(n)| = \sup_{n \in \mathbb{N}} |x_n| = \|(x_n)_{n \in \mathbb{N}}\|_\infty$ ,

$\downarrow$

$$v(E)=0 \Leftrightarrow E=\emptyset$$

$$\text{so } \|(x_n)_{n \in \mathbb{N}}\|_\infty = \|x\|_\infty$$

$\hookrightarrow$  the  $L^\infty(\mathbb{N}, v)$ -norm of  $x \in L^\infty(\mathbb{N}, v)$ .

Prop: (This will allow us to see each  $f \in L^p(X, \mu)$  as a bounded operator :  $L^q \rightarrow \mathbb{R}$ ,  $\frac{1}{q} + \frac{1}{p} = 1$ ):

Let  $(X, \mu)$  be a measure space. Let  $1 \leq p \leq \infty$ . fix  $f \in L^p(X, \mu)$ .

Then, the map  $T_f : L^q(X, \mu) \rightarrow \mathbb{C}$  (or  $\mathbb{R}$ ) , where  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$g \mapsto \int_X f g \, d\mu$$

is linear and bounded, with  $\|T_f\| = \|f\|_p$ .

Proof:  $T_f$  linear, and, for all  $g \in L^q(X, \mu)$ ,  $|T_f(g)| = \left| \int_X f g \, d\mu \right| \leq \underbrace{\|f\|_p}_{\text{fixed constant}} \cdot \|g\|_q$ ,

so  $T_f$  bounded and  $\|T_f\| \leq \|f\|_p$ .

In fact,  $\|T_f\| = \|f\|_p$ , because:

- for  $1 \leq p < +\infty$ , the function  $g_0 := \frac{1}{\|f\|_p^{p-1}} \cdot \frac{|f|^p}{f} \cdot \chi_{\{x \in X : f(x) \neq 0\}}$

belongs to  $L^q(X, \mu)$ , has  $\|g_0\|_q = 1$  and is s.t.  $T_f(g_0) = \|f\|_p$

$$\text{for } q \neq +\infty, \int_X |g_0|^q d\mu = \frac{1}{\|f\|_p^{q(p-1)}} \cdot \int_{\{x : f(x) \neq 0\}} \frac{|f|^{p \cdot q}}{|f|^q} d\mu =$$

$$\begin{aligned} T_f(g_0) &= \int_X f \cdot g_0 d\mu = \\ &= \frac{1}{\|f\|_p^{p-1}} \cdot \int_{\{x : f(x) \neq 0\}} f \cdot \frac{|f|^p}{f} d\mu = \\ &= \frac{1}{\|f\|_p^{p-1}} \cdot \int_X |f|^p d\mu = \frac{\|f\|_p^p}{\|f\|_p^{p-1}} = \|f\|_p. \end{aligned}$$

$$q(p-1) = p \quad \text{so} \quad \frac{1}{\|f\|_p^p} \cdot \int_{\{x : f(x) \neq 0\}} |f|^p d\mu = \frac{\|f\|_p^p}{\|f\|_p^p} = 1,$$

and for  $q = +\infty$ ,  $p-1=0$ , so  $g_0 = \frac{|f|}{f} \cdot \chi_{\{x \in X : f(x) \neq 0\}}$ ,

so  $\|g_0\|_\infty = \text{essential supremum of } \left| \frac{|f|}{f} \right| = 1$ .

- For  $p=+\infty$ , there exists a sequence  $(g_n)_{n \in \mathbb{N}}$  of functions  $g_n \in L^1(X, \mu)$ , with  $\|g_n\|_1 \leq 1$  and  $T_f(g_n) \rightarrow \|f\|_\infty$ . Can you show it? (Hint: essentially, it is a generalisation of what happens in the special case where  $f \in l^\infty$ ).



How does the above translate in  $l^p$  ( $= L^p(\mathbb{N}, \nu)$ )? Like this:

Let  $1 \leq p \leq +\infty$ , and  $(x_n)_{n \in \mathbb{N}} \in l^p$ . Then, the map

$$T_{(x_n)_{n \in \mathbb{N}}} : l^q \longrightarrow \mathbb{C} \text{ (or } \mathbb{R})$$

$$(y_n)_{n \in \mathbb{N}} \longmapsto \sum_{n=1}^{+\infty} x_n y_n \quad \left( = \int_{\mathbb{N}} x(n) y(n) d\nu(n) \right), \text{ where } \frac{1}{p} + \frac{1}{q} = 1,$$

is well-defined, linear, bounded, and  $\|T_{(x_n)_{n \in \mathbb{N}}}\| = \|(x_n)_{n \in \mathbb{N}}\|_p$ .

Corollary: Let  $1 \leq p \leq +\infty$ , and  $f \in L^p(X, \mu)$ . Then,  $\|f\|_p = \sup_{\|g\|_q=1} \int_X f g d\mu$ .

$$\text{Proof: } \|f\|_p = \|T_f\| = \sup_{\|g\|_q=1} |T_f g| = \sup_{\|g\|_q=1} \left| \int_X f g d\mu \right| = \sup_{\|g\|_q=1} \int_X f g d\mu.$$

Facts: Let  $(X, \mu)$  be a measure space.

- for  $1 \leq p \leq +\infty$ ,  $(L^p(X, \mu))^* = L^q(X, \mu)$ , for  $\frac{1}{q} + \frac{1}{p} = 1$ .

In particular, each  $T \in (L^p(X, \mu))^*$  is of the form  $T_g$ , for some  $g \in L^q(X, \mu)$ ,

i.e., for each  $T: L^p(X, \mu) \xrightarrow[\text{(or } \mathbb{R})]{} \mathbb{C}$  linear and bounded,

$\exists g \in L^q(X, \mu)$  s.t.  $T = T_g$ , i.e.

$$T(f) = \int_X f g d\mu, \text{ for all } f \in L^p(X, \mu).$$

- $(L^1(X, \mu))^* = L^\infty(X, \mu)$  when  $X$  is  $\sigma$ -finite w.r.t.  $\mu$ ,

i.e. when  $\exists X_1 \subseteq X_2 \subseteq \dots \subseteq X_n \subseteq \dots X$  s.t.

$$X = \bigcup_{n=1}^{+\infty} X_n \text{ and } \mu(X_n) < +\infty, \text{ for all } n \in \mathbb{N}.$$

More precisely, when  $X$  is  $\sigma$ -finite, for each  $T \in (L^1(X, \mu))^*$   $\exists f \in L^\infty(X, \mu)$ ,

s.t.  $T = T_f$ , i.e.  $T(g) = T_f(g)$ ,  $\forall g \in L^1(X, \mu)$ .

However, when  $X$  is not  $\sigma$ -finite, not every  $T \in (L^1(X, \mu))^*$  is necessarily equal to  $T_f$ , for some  $f \in L^\infty(X, \mu)$ .

- $(L^\infty(X, \mu))^* \supsetneq L^1(X, \mu)$  in general.

More precisely, in general there exist elements  $T \in (L^\infty(X, \mu))^*$  that are not equal to  $T_f$ , for any  $f \in L^1(X, \mu)$ .

- Corollary:
- for  $1 \leq p \leq +\infty$ ,  $\left(L^p(\mathbb{R}^n, m)\right)^* = L^q(\mathbb{R}^n, m)$ , where  $\frac{1}{q} + \frac{1}{p} = 1$ ,  
where  $m$  is the Lebesgue measure.
  - for  $1 \leq p \leq +\infty$ ,  $(l^p)^* = l^q$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ .

Proof: for  $1 \leq p \leq +\infty$ , the statement holds in any measure space  $(X, \mu)$ .  
 for  $p=1$ , the statement holds because  $\mathbb{R}^n$  is  $\sigma$ -finite w.r.t.  $m$ ,  
 and  $\mathbb{N}$  is  $\sigma$ -finite w.r.t.  $v$ .



Sidenote (not for exam):

In your Linear Analysis course, you saw the following corollary of the Hahn-Banach theorem: In a normed space  $(X, \| \cdot \|)$ ,  $\|x\| = \sup_{\substack{f \in X^*, \\ \|f\|=1}} |f(x)|$ .

for  $f \in L^p(X, \mu)$ , where  $(X, \mu)$  is a measure space and  $1 \leq p \leq \infty$ , this means that

$$\|f\|_p = \sup_{\substack{T \in (L^p)^* \\ \|T\|=1}} |T(f)|. \quad ①$$

for  $1 \leq p \leq \infty$ ,  $\{T \in (L^p)^* : \|T\|=1\} = \{T_g : \|g\|_q=1\}$  (we have explained this),

$$\text{so } \|f\|_p = \sup_{\|g\|_q=1} |T_g(f)| = \sup_{\|g\|_q=1} \left| \int_X f g \, d\mu \right| = \sup_{\|g\|_q=1} \int_X f g \, d\mu \text{ (which we have also explained).}$$

However, for  $p=1$  or  $p=\infty$ ,  $(L^1(X, \mu))^*$  may not be  $L^\infty(X, \mu)$ ,  
and  $(L^\infty(X, \mu))^*$  is usually  $\neq L^1(X, \mu)$ .

$$\text{So, the fact that } \|f\|_1 = \sup_{\|g\|_\infty=1} \int_X f g \, d\mu$$

$$\text{and } \|f\|_\infty = \sup_{\|g\|_1=1} \int_X f g \, d\mu$$

is a surprise, in the sense  
that ① says that  $\|f\|_p$  is the  
supremum of a larger set.

Exercise: Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{C}$  (or  $\mathbb{R}$ ). Fix  $1 \leq p \leq \infty$ .

Suppose that  $\sum_{n=1}^{+\infty} x_n y_n$  converges, for all  $(y_n)_{n \in \mathbb{N}}$  in  $\ell^q$  (where  $\frac{1}{q} + \frac{1}{p} = 1$ ).

Then,  $(x_n)_{n \in \mathbb{N}} \in \ell^p$ .

Solution: Fix  $1 \leq p \leq \infty$ .  $(x_n)_{n \in \mathbb{N}} \in \ell^p \iff \sup \left\{ \sum_{n=1}^k |x_n|^p : k \in \mathbb{N} \right\} < \infty \iff \sup \left\{ \left( \sum_{n=1}^k |x_n|^p \right)^{1/p} : k \in \mathbb{N} \right\} < \infty.$

[Idea: For each  $k$ , we will express  $\left( \sum_{n=1}^k |x_n|^p \right)^{1/p}$  as the norm of a linear map, and use UBP.]

For each  $k$ ,  $\left( \sum_{n=1}^k |x_n|^p \right)^{1/p} = \|x^{(k)}\|_p$ , where  $x^{(k)} = (x_1, x_2, \dots, x_k, 0, 0, \dots) \in \ell^p$ .

And we have explained that  $\|x^{(k)}\|_p = \|T_{x^{(k)}}\|$ ,

where  $T_{x^{(k)}} : \ell^q \rightarrow \mathbb{C}$   
 $\text{(or } \mathbb{R})$

$$(y_n)_{n \in \mathbb{N}} \mapsto \underbrace{\sum_{n=1}^{+\infty} x_n^{(k)} y_n}_{= \sum_{n=1}^k x_n^{(k)} y_n} = \int_{\mathbb{N}} x^{(k)}(n) y(n) d\nu(n)$$

- So:
- for each  $k \in \mathbb{N}$ ,  $T_{x^{(k)}} : \ell^q \rightarrow \mathbb{R}$  is linear and bounded (with  $\|T_{x^{(k)}}\| = \|x^{(k)}\|_p$ ).
  - $\ell^q$  is Banach.
  - for each  $y = (y_n)_{n \in \mathbb{N}} \in \ell^q$ ,  $T_{x^{(k)}}(y) = \sum_{n=1}^k x_n y_n$ ,
- so  $\{T_{x^{(k)}}(y) : k \in \mathbb{N}\} = \{\sum_{n=1}^k x_n y_n : k \in \mathbb{N}\}$  bounded in  $\mathbb{R}$   
(or  $\mathbb{C}$ ),
- as  $\left(\sum_{n=1}^k x_n y_n\right)_{k \in \mathbb{N}}$  is a convergent sequence  
(it converges to  $\sum_{n=1}^{\infty} x_n y_n$  by our assumptions).

Therefore, by UBP,  $\{T_{x^{(k)}} : k \in \mathbb{N}\}$  is bounded in  $(\ell^q)^*$ , i.e.

$$\underbrace{\{\|T_{x^{(k)}}\| : k \in \mathbb{N}\}}_{\|x^{(k)}\|_p} \text{ is bounded, i.e. } \left\{\left(\sum_{n=1}^k |x_n|^p\right)^{1/p} : k \in \mathbb{N}\right\} \text{ is bounded, i.e. } \sum_{n=1}^{\infty} |x_n|^p < \infty,$$

i.e.  $(x_n)_{n \in \mathbb{N}} \in \ell^p$ .

## Lecture 7 (21/10/2014)

In the previous lecture, we saw the following:

Let  $1 \leq p \leq \infty$ . Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\ell$ . Suppose that

$\sum_{n=1}^{\infty} x_n y_n$  converges, for all  $(y_n)_{n \in \mathbb{N}} \in \ell^q$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ . Then,  $(x_n)_{n \in \mathbb{N}} \in \ell^p$ .

This generalises in any  $\sigma$ -finite measure space. More precisely:

Thm: Let  $(X, \mu)$  be a  $\sigma$ -finite measure space. Let  $f: X \rightarrow \mathbb{R}$  be a measurable  
(or  $\mathbb{C}$ )

function. Suppose that  $\int_X f \cdot g \, d\mu$  is well-defined and finite, for all  $g \in L^q(X, \mu)$ ,

where  $\frac{1}{p} + \frac{1}{q} = 1$ . Then,  $f \in L^p(X, \mu)$ .



Note that  $(\mathbb{N}, \text{counting measure})$  and  $(\mathbb{R}^n, \text{Lebesgue measure})$  are  $\sigma$ -finite.

The proof is a careful generalisation of the proof in  $\ell^p = L^p(\mathbb{N}, \nu)$  (using UBP).  
↓  
counting measure

Open Mapping Theorem: Let  $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$  be both Banach spaces.

Let  $T: (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$  be linear, bounded and surjective.

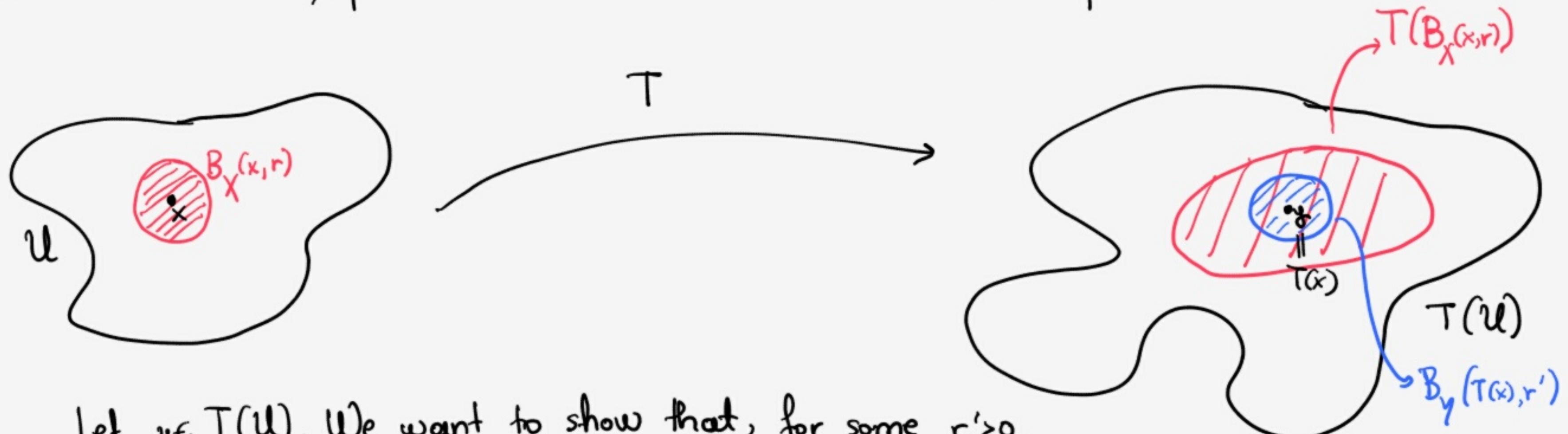
Then,  $T$  is an open map, i.e. sends open sets to open sets.



Note that any linear map  $T: (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$  that is bounded inverts open sets to open sets, because it is continuous. With the extra hypotheses that  $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$  are Banach and  $T$  is surjective, we have the inverse too, i.e. that  $T$  sends open sets to open sets.

## Proof of Open Mapping Theorem:

Idea: Let  $U \subseteq X$ , open. We want to show that  $T(U)$  is open in  $Y$ .



Let  $y \in T(U)$ . We want to show that, for some  $r' > 0$ ,

$B_y(y, r') \subseteq T(U)$ . Now,  $y \in T(U)$ , so  $y = T(x)$ , for some  $x \in U$ .

$U$  open  $\subseteq X$ , so  $\exists r > 0$  s.t.  $B_X(x, r) \subseteq U$ . Thus,  $T(B_X(x, r)) \subseteq T(U)$ .

So, it is enough to show that, for some  $r' > 0$ ,  $B_y(y, r') \subseteq T(B_X(x, r))$ .

$$B_y(y, r') \quad ||$$

The idea is to show this for  $B_X(0,1)$ ; then it should be easy to show it for any  $B_X(x,r)$ .

Thus, the proof of the Open Mapping theorem will be done in 3 steps:

Step 1:  $T(B_X(0,1)) \supseteq B_Y(0, \delta_0)$ , for some  $\delta_0 > 0$ .

Step 2:  $T(B_X(0,1)) \supseteq B_Y(0, \delta_0/2)$ .

Step 3:  $U \subseteq X$  open  $\implies T(U)$  open in  $Y$ .

Let us see the proof of Step 1. The ingredients are:  $(Y, \|\cdot\|_Y)$  Banach,  $T$  surjective and linear:

$$X = \bigcup_{n=1}^{+\infty} B_X(0,n) \implies T(X) = T\left(\bigcup_{n=1}^{+\infty} B_X(0,n)\right) = \bigcup_{n=1}^{+\infty} T(B_X(0,n)) = \bigcup_{n=1}^{+\infty} \overline{T(B_X(0,n))}.$$

$\Downarrow$   $\begin{matrix} Y \\ \text{surjective} \end{matrix}$

$$\text{So, } Y = \bigcup_{n=1}^{+\infty} \overline{T(B_X(0,n))}^0.$$

$\begin{matrix} \text{closed} \\ \text{Banach} \end{matrix}$

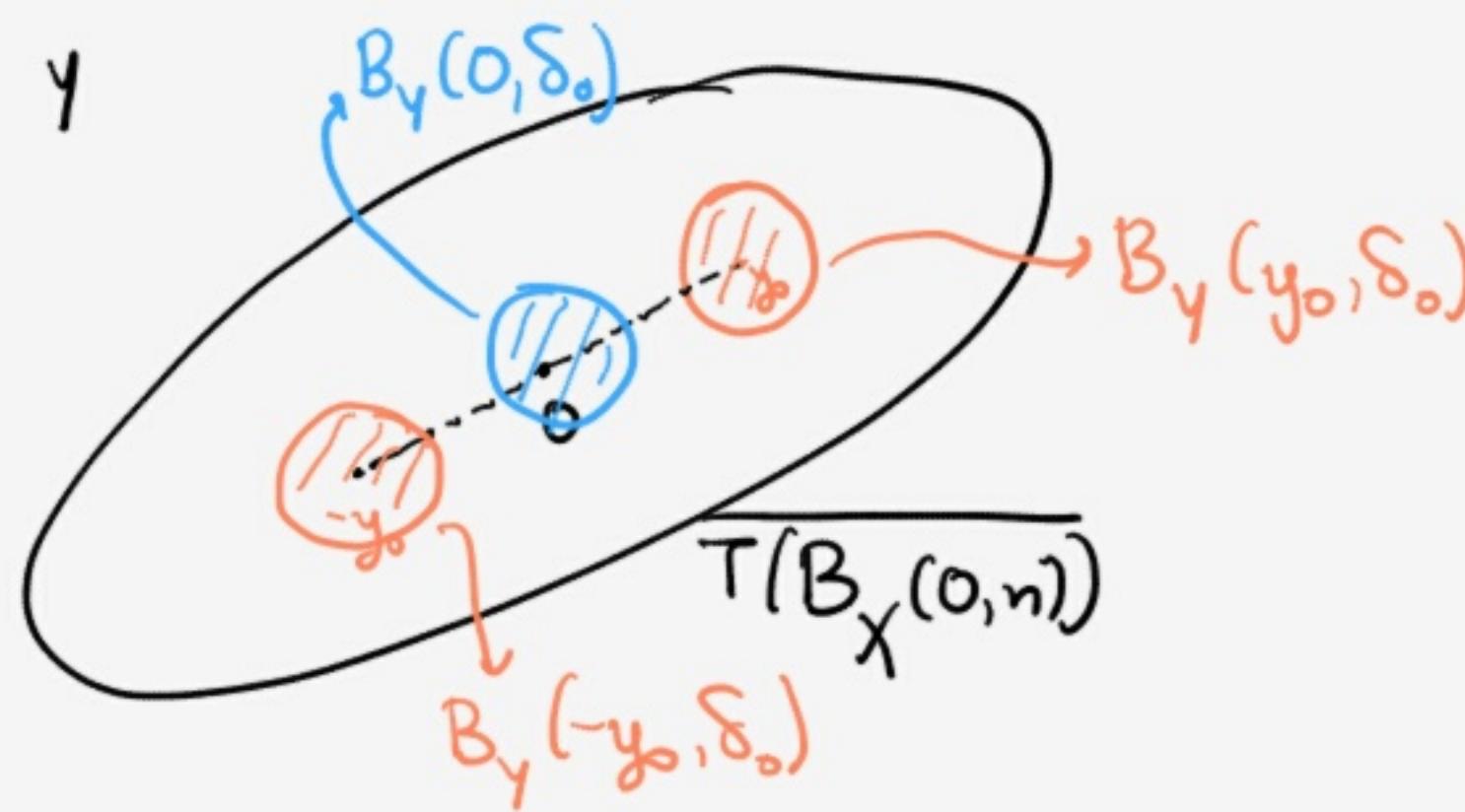
By Baire's theorem,  $\exists n_0 \in \mathbb{N}: T(B_X(0,n_0)) \neq \emptyset$ ,

i.e.  $\exists y_0 \in Y$  and  $\delta_0 > 0$  s.t.  $\overline{T(B_X(0, n_0))} \supseteq B_Y(y_0, \delta_0)$ . Then,  $\overline{T(B_X(0, n_0))} \supseteq B_Y(0, \delta_0)$ :

the reason is that  $T(B_X(0, n))$  is convex and symmetric around 0

since  $T(B_X(0, n_0))$  is convex and symmetric around 0,  
as  $T$  is linear and  $B_X(0, n_0)$  is convex and symmetric around 0

$$\left. \begin{array}{l} \text{and thus : } B_Y(y_0, \delta_0) \subseteq \overline{T(B_X(0, n_0))} \\ \Rightarrow B_Y(-y_0, \delta_0) \subseteq \overline{T(B_X(0, n_0))} \end{array} \right\} \Rightarrow \frac{B_Y(-y_0, \delta_0) + B_Y(y_0, \delta_0)}{2} \subseteq \overline{T(B_X(0, n_0))}$$



$$\frac{B_Y(0, 2\delta_0)}{2} = B_Y(0, \delta_0).$$

(proof to be continued).

## Lecture 8 (23/10/2014)

Proof of Open Mapping Theorem (continued):

In the previous lecture, we had reached this point in **Step 1** of the proof:

$$\exists n_0 \in \mathbb{N}, \text{ s.t. } \overline{T(B_X(0, n_0))} \supseteq B_Y(0, r_0), \text{ for some } r_0 > 0.$$

$$\text{So, } \overline{T(B_X(0, 1))} = \frac{1}{n_0} \cdot \overline{T(B_X(0, n_0))} \supseteq \frac{1}{n_0} \cdot B_Y(0, r_0) = B_Y(0, \frac{r_0}{n_0}).$$

$$\text{Thus, } \exists \delta_0 > 0 \text{ s.t. } \overline{T(B_X(0, 1))} \supseteq B_Y(0, \delta_0).$$

Let us now move on to **Step 2** of the proof; we will show that

$$T(B_X(0, 1)) \supseteq B_Y\left(0, \frac{\delta_0}{2}\right) :$$

Let  $y \in B_Y\left(0, \frac{\delta_0}{2}\right)$ . We want to show that  $y = T(x)$ , for some  $x \in B_X(0, 1)$ .

Indeed, by Step 1,  $B_y(0, \frac{\delta_0}{2}) \subseteq \overline{T(B_X(0, \frac{1}{2}))}$ ,

$$\text{so } y \in \overline{T(B_X(0, \frac{1}{2}))},$$

so  $\exists x_1 \in B_X(0, \frac{1}{2})$ , s.t.  $\|y - T(x_1)\| < \frac{\delta_0}{2^2}$

$$\text{i.e., } \|x_1\| < \frac{1}{2}$$

$$\text{i.e., } y - T(x_1) \in B_y(0, \frac{\delta_0}{2^2})$$

By Step 1,  $B_y(0, \frac{\delta_0}{2^2}) \subseteq \overline{T(B_X(0, \frac{1}{2^2}))}$ ,

$$\text{so } y - T(x_1) \in \overline{T(B_X(0, \frac{1}{2^2}))},$$

so  $\exists x_2 \in B_X(0, \frac{1}{2^2})$ , s.t.  $\|y - T(x_1) - T(x_2)\| < \frac{\delta_0}{2^3}$

$$\text{i.e., } \|x_2\| < \frac{1}{2^2}$$

$$\text{i.e., } y - T(x_1) - T(x_2) \in B_y(0, \frac{\delta_0}{2^3})$$

Continuing in this way, we have that

$$\forall n \in \mathbb{N}, \exists x_n \in B_X(0, \frac{1}{2^n}), \text{ s.t. } \|y - T(x_1) - T(x_2) - \dots - T(x_n)\| < \frac{\delta_0}{2^{n+1}}.$$

i.e.,  $\|x_n\| < \frac{1}{2^n}.$

It is obvious that, for the sequence  $(s_n)_{n \in \mathbb{N}}$  in  $X$  s.t.  $s_n = x_1 + \dots + x_n \quad \forall n \in \mathbb{N}$ , we have that  $T(s_n) \rightarrow y$ :

$$\|y - T(s_n)\| = \|y - (T(x_1) + T(x_2) + \dots + T(x_n))\| = \|y - T(x_1) - T(x_2) - \dots - T(x_n)\| < \frac{\delta_0}{2^{n+1}} \xrightarrow{n \rightarrow \infty} 0.$$

We will show that  $s_n \rightarrow x$ , for some  $x \in B_X(0, 1)$ . Then, we will be done with  
Step 2: By continuity of  $T$ , we will have that  $T(s_n) \rightarrow T(x)$ , and, by  
uniqueness of limits in metric spaces,  $y = T(x)$ . Well:

- $s_n \rightarrow x$ , for some  $x \in X$ :

$(s_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $X$ , as, for all  $n > m$ ,

$$\|s_n - s_m\| = \|x_{m+1} + \dots + x_n\| \leq \|x_{m+1}\| + \dots + \|x_n\| < \frac{1}{2^{m+1}} + \dots + \frac{1}{2^n} \xrightarrow[m, n \rightarrow \infty]{} 0.$$

So, since  $X$  is a Banach space,  $\exists x \in X$  s.t.  $s_n \rightarrow x$ .

- $x \in B_X(0, 1)$ :

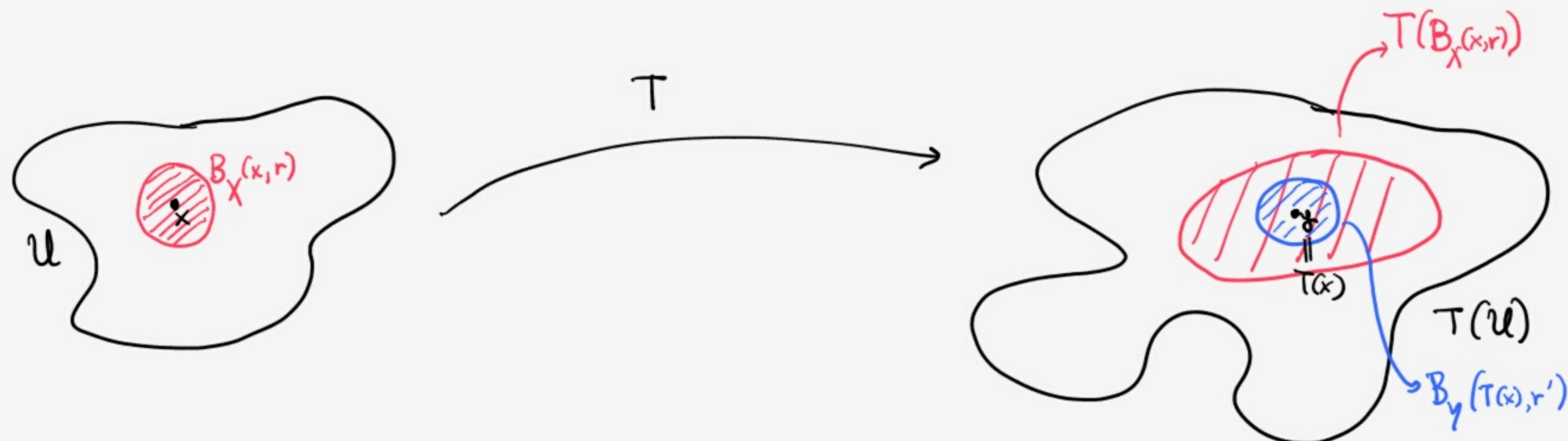
for all  $n \in \mathbb{N}$ ,  $\|s_n\| = \|x_1 + \dots + x_n\| \leq \|x_1\| + \|x_2\| + \dots + \|x_n\| < \|x_1\| + \sum_{k=2}^n \frac{1}{2^k} < \|x_1\| + \frac{1}{2}$ ,

$$\text{so } \|x\| = \lim_{n \rightarrow \infty} \|s_n\| \leq \|x_1\| + \frac{1}{2} \leq \frac{1}{2} + \frac{1}{2} = 1.$$

So, Step 2 is complete (note that the ingredients in this step were that  $T$  is linear and continuous, and that  $X$  is Banach).

We now finish by completing Step 3 :  $U \subseteq X$  open  $\implies T(U)$  open in  $Y$ .

Indeed, let  $U \subseteq X$  open. Let  $y \in T(U)$ . We want to show that, for some  $r' > 0$ ,  $B_y(y, r') \subseteq T(U)$ . Indeed,  $y \in T(U)$ , so  $y = T(x)$ , for some  $x \in U$ .



$U$  open in  $X$ , so  $\exists r > 0$  s.t.  $B_X(x, r) \subseteq U$ .

$$\text{So, } \underbrace{T(B_X(x, r))}_{\parallel} \subseteq T(U)$$

$$T(x + r B_X(0, 1)) = T(x) + r \cdot T(B_X(0, 1)) \stackrel{\text{Step 2}}{\supseteq} T(x) + r \cdot B_y(0, \frac{\delta_0}{2}) = B_y(T(x), \frac{r \delta_0}{2}) = B_y(y, \frac{r \delta_0}{2}).$$

Thus,  $B_y(y, \frac{r\delta_0}{2}) \subseteq T(U)$ . Since  $y$  was an arbitrary element of  $T(U)$ , the proof is complete.

(Note that, in Step 3 of the proof, the only ingredients we used were the linearity of  $T$  and properties of norms).

The Open Mapping Theorem has an immediate corollary: that a continuous linear map between Banach spaces that is 1-1 and surjective is a homeomorphism:

Corollary: Let  $(X, \| \cdot \|_X)$ ,  $(Y, \| \cdot \|_Y)$  be Banach spaces. Let  $T: (X, \| \cdot \|_X) \rightarrow (Y, \| \cdot \|_Y)$  be linear, bounded, surjective and 1-1.

Then  $T^{-1}$  is linear and bounded.

Proof: Since  $T: X \rightarrow Y$  is 1-1 and surjective,  $T^{-1}: Y \rightarrow X$  is well-defined.

$T^{-1}$  is clearly linear. To show that  $T^{-1}$  is bounded, i.e. continuous, we will show that it inverts open sets to open sets.

Indeed, let  $U \subseteq X$  open. Then,

$$(T^{-1})^{-1}(U) = \{y \in Y : T^{-1}(y) \in U\} = \{y \in Y : y \in T(U)\} = T(U),$$

which is open in  $Y$ , as  $T$  is open, by the Open Mapping Theorem.

Using the above corollary of the Open Mapping Theorem, we will easily prove the Closed Graph Theorem. Like the corollary above, the Closed Graph Theorem ensures the continuity of some linear map under certain conditions.

Closed Graph Theorem: Let  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  both be Banach spaces.

Let  $T: X \rightarrow Y$  be a linear map. Suppose that the graph of  $T$  is closed in  $(X \times Y, \|\cdot\|_{X \times Y})$ . Then,  $T$  is bounded.



We define  $\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$ , for all  $(x, y) \in X \times Y$ . It is very easy to see that  $(X \times Y, \|\cdot\|_{X \times Y})$  is a normed space.

Moreover, by the definition of  $\|\cdot\|_{X \times Y}$ , we have that, for a sequence  $((x_n, y_n))_{n \in \mathbb{N}}$  in  $X \times Y$ ,  $(x_n, y_n) \xrightarrow[\|\cdot\|_{X \times Y}]{} (x, y) \iff x_n \xrightarrow[\|\cdot\|_X]{} x \text{ and } y_n \xrightarrow[\|\cdot\|_Y]{} y$ ,

and  $((x_n, y_n))_{n \in \mathbb{N}}$  Cauchy in  $(X \times Y, \|\cdot\|_{X \times Y}) \iff ((x_n)_{n \in \mathbb{N}} \text{ Cauchy in } (X, \|\cdot\|_X) \text{ and } (y_n)_{n \in \mathbb{N}} \text{ Cauchy in } (Y, \|\cdot\|_Y))$ .

Therefore, if  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  are Banach, then  $(X \times Y, \|\cdot\|_{X \times Y})$  is Banach.

Note that this is really why we use this particular norm on  $X \times Y$ : so that behaviour in  $X \times Y$  is dictated by behaviour in  $X$  and in  $Y$ .

## Lecture 9 (28/10/2014)

Closed Graph Theorem: Let  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  be Banach spaces, and  $T: X \rightarrow Y$  a linear map. If the graph  $\Gamma(T)$  of  $T$  is closed in  $(X \times Y, \|\cdot\|_{X \times Y})$ , then  $T$  is bounded.

Observation: The Closed Graph Theorem simplifies a lot the usual criterion to check continuity via convergence of sequences (of course, only when the conditions of the theorem are satisfied). More precisely:

When  $T: X \rightarrow Y$  is a function between the normed spaces  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$ , then it is continuous if:

whenever  $(x_n)_{n \in \mathbb{N}}$  in  $X$  is s.t.  $x_n \xrightarrow{\|\cdot\|_X} x$  for some  $x \in X$ , then  $T(x_n) \xrightarrow{\|\cdot\|_Y} T(x)$ .

However, when we also know that  $T$  is linear and  $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$  are both Banach spaces, then, by the Closed Graph Theorem,  $T$  is continuous if

the graph  $\Gamma(T)$  of  $T$  is closed in  $(X \times Y, \|\cdot\|_{X \times Y})$ ,

i.e. whenever  $\left( (x_n, T(x_n)) \right)_{n \in \mathbb{N}}$  in  $\Gamma(T)$  is s.t.  $(x_n, T(x_n)) \xrightarrow[\|\cdot\|_{X \times Y}]{} (x, y)$  for some  $(x, y) \in X \times Y$ , then  $(x, y) \in \Gamma(T)$  ( $\Leftrightarrow y = T(x)$ ).

i.e. whenever  $(x_n)_{n \in \mathbb{N}}$  in  $X$  is s.t.  $x_n \xrightarrow[\|\cdot\|_X]{} x$  for some  $x \in X$  and  $T(x_n) \xrightarrow[\|\cdot\|_Y]{} y$  for some  $y \in Y$ , then  $y = T(x)$ ,

i.e. whenever  $(x_n)_{n \in \mathbb{N}}$  in  $X$  is s.t.  $x_n \xrightarrow[\|\cdot\|_X]{} x$  for some  $x \in X$  [and  $T(x_n)$  converges in  $Y$ ],  
 then  $T(x_n) \xrightarrow[\|\cdot\|_Y]{} T(x)$ .  
 ↓  
 not in original criterion!

This means that, when we try to show that a map  $T: X \rightarrow Y$  is continuous when the conditions of the Closed Graph Theorem are satisfied, we can start the usual way:

take  $(x_n)_{n \in \mathbb{N}}$  in  $X$  with  $x_n \xrightarrow[\| \cdot \|_X]{} x$  for some  $x \in X$ , and show that  $T(x_n) \xrightarrow[\| \cdot \|_Y]{} T(x)$ ,

but this time assuming that  $T(x_n)$  converges in  $Y$ ! Normally, we would show

- (a) that  $T(x_n)$  converges, and
- (b) that its limit is  $T(x)$ .

Under the conditions of the Closed Graph Theorem, we are allowed to assume

(a), and we only need to show (b). And assuming (a) usually simplifies things a lot.

Proof of the Closed Graph Theorem:  $\Gamma(T) = \{(x, T(x)) : x \in X\}$  is a subspace of  $X \times Y$ .

$$\begin{aligned} \text{Indeed, } & (x_1, T(x_1)), (x_2, T(x_2)) \in \Gamma(T), (x_1, T(x_1)) + (x_2, T(x_2)) = (x_1 + x_2, T(x_1) + T(x_2)) = \\ & = (x_1 + x_2, T(x_1 + x_2)) \in \Gamma(T), \end{aligned}$$

and if scalars  $\lambda$  and  $t(x, T(x)) \in \Gamma(T)$ ,  $\lambda \cdot (x, T(x)) = (\lambda x, \lambda T(x)) = (x, T(\lambda x)) \in \Gamma(T)$ .

Note that for this we have only used that  $T$  is linear.

Now,  $(X, \| \cdot \|_X)$ ,  $(Y, \| \cdot \|_Y)$  are Banach, so  $(X \times Y, \| \cdot \|_{X \times Y})$  is Banach.

So,  $\Gamma(T)$  is a closed subspace of the Banach space  $(X \times Y, \| \cdot \|_{X \times Y})$ , so  
by assumption

$(\Gamma(T), \| \cdot \|_{X \times Y})$  is Banach.

$$\begin{array}{ccc} & \nearrow \text{Banach} & \searrow \text{Banach} \\ \text{We define the projection } P : (\Gamma(T), \| \cdot \|_{X \times Y}) & \longrightarrow & (X, \| \cdot \|_X) \\ (x, T(x)) & \longrightarrow & x \end{array}$$

$P$  is clearly linear, surjective and 1-1. It is also bounded:

$$\|P(x, T(x))\|_X = \|x\|_X \leq \|x\|_X + \|T(x)\|_Y = \|(x, T(x))\|_Y, \quad \forall (x, T(x)) \in \Gamma(T).$$

By the Corollary of the Open Mapping Theorem in the previous lecture,

$$P^{-1} : (X, \|\cdot\|_X) \longrightarrow (T(T), \|\cdot\|_{X \times Y})$$

$x \quad \longrightarrow \quad (x, T(x))$

is bounded, i.e. if  $C \geq 0$  s.t.

$$\|P^{-1}(x)\|_{X \times Y} \leq C \cdot \|x\|_X, \quad \forall x \in X.$$

$$\|\cdot\|_{X \times Y}$$

$$\|\cdot\|_X + \|T(x)\|_Y$$

$$\text{So, } \underbrace{\|x\|_X + \|T(x)\|_Y}_{\|T(x)\|_Y} \leq C \cdot \|x\|_X, \text{ so } \|T(x)\|_Y \leq C \cdot \|x\|_X, \quad \forall x \in X.$$

$$\|T(x)\|_Y$$

Thus,  $T$  is bounded.

App: Showing that a space  $(X, \|\cdot\|_X)$  is not Banach.

A way to show that  $(X, \|\cdot\|_X)$  is not Banach is to construct a linear map  $T: (X, \|\cdot\|_X) \longrightarrow (Y, \|\cdot\|_Y)$  for a Banach space  $(Y, \|\cdot\|_Y)$ , s.t. the graph of  $T$  is closed but  $T$  is not bounded. This will mean that the conditions of the Closed Graph Theorem are not satisfied. So,  $(X, \|\cdot\|_X)$  cannot be Banach.

ex:  $(C^1([0,1]), \|\cdot\|_\infty)$  is not Banach. Can you show it?

[Note that  $C^1([0,1]) := \{f: [0,1] \rightarrow \mathbb{R} : f \text{ differentiable and } f' \text{ continuous}\}$ .]  
[And, while  $(C([0,1]), \|\cdot\|_\infty)$  is Banach,  $(C^1([0,1]), \|\cdot\|_\infty)$  is not.]

And so, the functional Analysis part of this course has come to an end.

for anyone who is interested, the thing to study next is weak topologies.

More particularly, the idea is to understand a normed space  $X$  through its functionals, i.e. the space  $X^* = \{ f: X \rightarrow \mathbb{R} \text{ linear and bounded} \}$ .

(we have seen an expression of this idea, via the fact that  
 $K \subseteq X$  is bounded  $\iff f(K)$  is bounded,  $\forall f \in X^*$  ).

To achieve the above, we consider the smallest topology on  $X$  s.t. each  $f \in X^*$  is still a continuous function, i.e. the smallest topology on  $X$  that contains the inverse images of all open subsets of  $\mathbb{R}$  through all  $f \in X^*$ .

We call it the weak topology, because it is generally smaller than the

topology induced by the norm on  $X$ .

The fact that the weak topology is so small allows some more freedom.

for example, in many normed spaces, closed balls are compact w.r.t. the weak topology (there are fewer open covers w.r.t. the weak topology, and thus it is more likely for each of them to have an open subcover).

So, the use of the weak topology reveals some structure in  $X$ , which was hidden when we were using the topology induced by the norm.

This allows us to understand  $X$  more deeply.

Weak topologies have applications in measure theory, and thus in Analysis and Probability.